Round 4
www.artofproblemsolving.com/community/c2382160
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– Day 1

Problem 1 Let $f(x)=x^n+a_1x^{n-1}+\ldots+a_n$ $(n\ge3)$ be a polynomial with real coefficients and n real roots, such that $\frac{a_{n-1}}{a_n}>n+1$. Prove that if $a_{n-2}=0$, then at least one root of $f(x)$ lies in the open interval $\left(-\frac12,\frac1{n+1}\right)$.

Problem 2 Let there be given a polygon P which is mapped onto itself by two rotations: ρ_1 with center O_1 and angle ω_1, and ρ_2 with center O_2 and angle ω_2 $(0<\omega_i<2\pi)$. Show that the ratio $\frac{\omega_1}{\omega_2}$ is rational.

Problem 3 Let $MABCD$ be a pyramid with the square $ABCD$ as the base, in which $MA=MD$, $MA^2+AB^2=MB^2$ and the area of $\triangle ADM$ is equal to 1. Determine the radius of the largest ball that is contained in the given pyramid.

– Day 2

Problem 4 The sequence $(x_n)_{n\in\mathbb N}$ is defined by $x_1=x_2=1$, $x_{n+2}=14x_{n+1}-x_n-4$ for each $n\in\mathbb N$. Prove that all terms of this sequence are perfect squares.

Problem 5 Let E be a point on the median AD of a triangle ABC, and F be the projection of E onto BC. From a point M on EF the perpendiculars MN to AC and MP to AB are drawn. Prove that if the points N,E,P lie on a line, then M lies on the bisector of $\angle BAC$.

Problem 6 Let Δ be the set of all triangles inscribed in a given circle, with angles whose measures are integer numbers of degrees different than $45^\circ,90^\circ$ and 135°. For each triangle $T\in\Delta$, $f(T)$ denotes the triangle with vertices at the second intersection points of the altitudes of T with the circle.

(a) Prove that there exists a natural number n such that for every triangle $T\in\Delta$, among the triangles $T,f(T),\ldots,f^n(T)$ (where $f^0(T)=T$ and $f^k(T)=f(f^{k-1}(T))$) at least two are equal.
(b) Find the smallest n with the property from (a).