Round 4

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– Day 1

Problem 1 Find all real parameters q for which there is a $p \in [0,1]$ such that the equation

$$x^4 + 2px^3 + (2p^2 - p)x^2 + (p-1)p^2x + q = 0$$

has four real roots.

Problem 2 Let n and k be natural numbers and p a prime number. Prove that if k is the exact exponent of p in $2^{2^n} + 1$ (i.e. p^k divides $2^{2^n} + 1$, but p^{k+1} does not), then k is also the exact exponent of p in $2^{p-1} - 1$.

Problem 3 Let M be an arbitrary interior point of a tetrahedron $ABCD$, and let S_A, S_B, S_C, S_D be the areas of the faces BCD, ACD, ABD, ABC, respectively. Prove that

$$S_A \cdot MA + S_B \cdot MB + S_C \cdot MC + S_D \cdot MD \geq 9V,$$

where V is the volume of $ABCD$. When does equality hold?

– Day 2

Problem 4 Let A, B, C be non-collinear points. For each point D of the ray AC, we denote by E and F the points of tangency of the incircle of $\triangle ABD$ with AB and AD, respectively. Prove that, as point D moves along the ray AC, the line EF passes through a fixed point.

Problem 5 The points of space are painted in two colors. Prove that there is a tetrahedron such that all its vertices and its centroid are of the same color.

Problem 6 Find all polynomials $p(x)$ satisfying $p(x^3 + 1) = p(x+1)^3$ for all x.