Round 4

www.artofproblemsolving.com/community/c2383846

by jasperE3, Arman, hsiangshen

– Day 1

Problem 1 Determine all natural numbers n for which there exists a permutation $(a_1, a_2, \dots, a_n)$ of the numbers $0, 1, \dots, n-1$ such that, if b_i is the remainder of $a_1 a_2 \cdots a_i$ upon division by n for $i = 1, \dots, n$, then $(b_1, b_2, \dots, b_n)$ is also a permutation of $0, 1, \dots, n-1$.

Problem 2 Let $b_1 \ge b_2 \ge \dots \ge b_n$ be nonnegative numbers, and $(a_1, a_2, \dots, a_n)$ be an arbitrary permutation of these numbers. Prove that for every $t \ge 0$,

$$(a_1a_2 + t)(a_3a_4 + t) \cdots (a_{2n-1}a_{2n} + t) \le (b_1b_2 + t)(b_3b_4 + t) \cdots (b_{2n-1}b_{2n} + t).$$

Problem 3 A regular triangular pyramid $ABCD$ with the base side $AB = a$ and the lateral edge $AD = b$ is given. Let M and N be the midpoints of AB and CD respectively. A line α through MN intersects the edges AD and BC at P and Q, respectively.

(a) Prove that $AP/AD = BQ/BC$.
(b) Find the ratio AP/AD which minimizes the area of $MQNP$.

– Day 2

Problem 4 Find the smallest possible side of a square in which five circles of radius 1 can be placed, so that no two of them have a common interior point.

Problem 5 Can the polynomials $x^5 - x - 1$ and $x^2 + ax + b$, where $a, b \in Q$, have common complex roots?

Problem 6 Let $a, b, c > 0$ satisfy for all integers n, we have

$$\lfloor an \rfloor + \lfloor bn \rfloor = \lfloor cn \rfloor$$

Prove that at least one of a, b, c is an integer.