Round 4

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– Day 1

Problem 1 Find all pairs of natural numbers (n,k) for which
$(n+1)^k-1=n!$.

Problem 2 Let n unit circles be given on a plane. Prove that on one of the circles there is an arc of length at least $\frac{2\pi}{n}$ not intersecting any other circle.

Problem 3 In a regular $2n$-gonal prism, bases $A_1A_2\cdots A_{2n}$ and $B_1B_2\cdots B_{2n}$ have circumradii equal to R. If the length of the lateral edge A_1B_1 varies, the angle between the line A_1B_{n+1} and the plane $A_1A_3B_{n+2}$ is maximal for $A_1B_1=2R\cos\frac{\pi}{2n}$.

– Day 2

Problem 4 If $x_1,x_2,\ldots,x_n$ are arbitrary numbers from the interval $[0,2]$, prove that

$$\sum_{i=1}^n\sum_{j=1}^n|x_i-x_j|\le n^2$$

When is the equality attained?

Problem 5 Find all values of parameters a,b for which the polynomial

$$x^4+(2a+1)x^3+(a-1)^2x^2+bx+4$$

can be written as a product of two monic quadratic polynomials $\Phi(x)$ and $\Psi(x)$, such that the equation $\Psi(x)=0$ has two distinct roots α,β which satisfy $\Phi(\alpha)=\beta$ and $\Phi(\beta)=\alpha$.

Problem 6 Find the locus of centroids of equilateral triangles whose vertices lie on sides of a given square $ABCD$.