Round 4

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– Day 1

Problem 1 Show that there are no integers x and y satisfying $x^2+5=y^3$.

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Problem 2 Points P, Q, R, S are taken on respective edges AC, AB, BD, and CD of a tetrahedron $ABCD$ so that PR and QS intersect at point N and PS and QR intersect at point M. The line MN meets the plane ABC at point L. Prove that the lines AL, BP, and CQ are concurrent.

Problem 3 Each side of a triangle ABC has been divided into $n+1$ equal parts. Find the number of triangles with the vertices at the division points having no side parallel to or lying at a side of $\triangle ABC$.

– Day 2

Problem 4 For each real number k, denote by $f(k)$ the larger of the two roots of the quadratic equation

$$(k^2+1)x^2+10kx-6(9k^2+1)=0.$$

Show that the function $f(k)$ attains a minimum and maximum and evaluate these two values.

Problem 5 A convex pentagon $ABCDE$ satisfies $AB=BC=CA$ and $CD=DE=EC$. Let S be the center of the equilateral triangle ABC and M and N be the midpoints of BD and AE, respectively. Prove that the triangles SME and SND are similar.

Problem 6 The set $M=\{1,2,\ldots,2n\}$ $(n\geq 2)$ is partitioned into k nonintersecting subsets $M_1, M_2, \ldots, M_k$, where $k^3+1\leq n$. Prove that there exist $k+1$ even numbers $2j_1, 2j_2, \ldots, 2j_{k+1}$ in M that are in one and the same subset M_j $(1\leq j\leq k)$ such that the numbers $2j_1-1, 2j_2-1, \ldots, 2j_{k+1}-1$ are also in one and the same subset M_r $(1\leq r\leq k)$.

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