Round 4

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by jasperE3, dgrozev

– Day 1

Problem 1 For natural number n and real numbers α and x satisfy the inequalities $\alpha^{n+1} \leq x \leq 1$ and $0 < \alpha < 1$. Prove that

$$\prod_{k=1}^{n} \left| \frac{x-\alpha^k}{x+\alpha^k} \right| \leq \prod_{k=1}^{n} \left| \frac{1-\alpha^k}{1+\alpha^k} \right|.$$

Borislav Boyanov

Problem 2 In the space are given n points and no four of them belongs to a common plane. Some of the points are connected with segments. It is known that four of the given points are vertices of tetrahedron which edges belong to the segments given. It is also known that common number of the segments, passing through vertices of tetrahedron is $2n$. Prove that there exists at least two tetrahedrons every one of which have a common face with the first (initial) tetrahedron.

N. Nenov, N. Hadzhiivanov

Problem 3 A given truncated pyramid has triangular bases. The areas of the bases are B_1 and B_2 and the area of the surface is S. Prove that if there exists a plane parallel to the bases whose intersection divides the pyramid to two truncated pyramids in which may be inscribed by spheres then

$$S = (\sqrt{B_1} + \sqrt{B_2})(\sqrt[4]{B_1} + \sqrt[4]{B_2})^2$$

G. Gantchev

– Day 2

Problem 4 Vertices A and C of the quadrilateral $ABCD$ are fixed points of the circle k and each of the vertices B and D is moving to one of the arcs of k with ends A and C in such a way that $BC = CD$. Let M be the intersection point of AC and BD and F is the center of the circumscribed circle around $\triangle ABM$. Prove that the locus of F is an arc of a circle.

J. Tabov

Problem 5 Let $Q(x)$ be a non-zero polynomial and k be a natural number. Prove that the polynomial $P(x) = (x-1)^k Q(x)$ has at least $k+1$ non-zero coefficients.

Problem 6 A Pythagorean triangle is any right-angled triangle for which the lengths of two legs and the length of the hypotenuse are integers. We are observing all Pythagorean triangles in which may be inscribed a quadrangle with sidelength integer number, two of which sides lie on the cathets and one of the vertices of which lies on the hypotenuse of the triangle given. Find the side lengths of the triangle with minimal surface from the observed triangles.

St. Doduneko