Round 4

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by jasperE3

– Day 1

Problem 1 Find all pairs of natural numbers (m, n) bigger than 1 for which $2^m + 3^n$ is the square of whole number.

I. Tonov

Problem 2 Let F be a polygon the boundary of which is a broken line with vertices in the knots (units) of a given in advance regular square network. If k is the count of knots of the network situated over the boundary of F, and ℓ is the count of the knots of the network lying inside F, prove that if the surface of every square from the network is 1, then the surface S of F is calculated with the formulae:

$$S = \frac{k}{2} + \ell - 1$$

V. Chukanov

Problem 3 Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ be a polynomial with real coefficients ($a_0 \neq 0$) such that $|f(x)| \leq 1$ for every $x \in [-1, 1]$. Prove that

(a) there exist a constant c (one and the same for all polynomials with the given property), for which
(b) $|a_0| \leq 4$.

V. Petkov

– Day 2

Problem 4 In the plane are given a circle k with radii R and the points $A_1, A_2, \ldots, A_n$, lying on k or outside k. Prove that there exist infinitely many points X from the given circumference for which

$$\sum_{i=1}^{n} A_iX^2 \geq 2nR^2.$$

Does there exist a pair of points on different sides of some diameter, X and Y from k, such that

$$\sum_{i=1}^{n} A_iX^2 \geq 2nR^2 \text{ and } \sum_{i=1}^{n} A_iY^2 \geq 2nR^2?$$

H. Lesov

Problem 5 Let the *subbishop* (a bishop is the figure moving only by a diagonal) be a figure moving only by diagonal but only in the next cells (squares) of the chessboard. Find the maximal count of subbishops over a chessboard $n \times n$, no two of which are not attacking.

V. Chukanov

Problem 6 Some of the faces of a convex polyhedron M are painted in blue, others are painted in white and there are no two walls with a common edge. Prove that if the sum of surfaces of the blue walls is bigger than half surface of M then it may be inscribed a sphere in the polyhedron given (M).

(H. Lesov)