Round 4

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by jasperE3

– Day 1

Problem 1 Let the sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by the conditions: $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ $(n = 1, 2, \ldots)$. Prove that:

(a) a_m and a_n are relatively prime numbers when $m \neq n$.
(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{a_k} = 1$

I. Tonov

Problem 2 Let the numbers a_1, a_2, a_3, a_4 form an arithmetic progression with difference $d \neq 0$. Prove that there are no exists geometric progressions b_1, b_2, b_3, b_4 and c_1, c_2, c_3, c_4 such that:

$$a_1 = b_1 + c_1, a_2 = b_2 + c_2, a_3 = b_3 + c_3, a_4 = b_4 + c_4.$$

Problem 3 Let $a_1, a_2, \ldots, a_n$ are different integer numbers in the range $[100, 200]$ for which: $a_1 + a_2 + \ldots + a_n \geq 11100$. Prove that it can be found at least number from the given in the representation of decimal system on which there are at least two equal digits.

L. Davidov

– Day 2

Problem 4 Find all functions $f(x)$ defined in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that are differentiable at 0 and satisfy

$$f(x) = \frac{1}{2}\left(1 + \frac{1}{\cos x}\right) f\left(\frac{x}{2}\right)$$

for every x in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

L. Davidov

Problem 5 Let the line ℓ intersects the sides AC, BC of the triangle ABC respectively at the points E and F. Prove that the line ℓ is passing through the incenter of the triangle ABC if and only if the following equality is true:

$$BC \cdot \frac{AE}{CE} + AC \cdot \frac{BF}{CF} = AB.$$

H. Lesov

Problem 6 In the tetrahedron $ABCD$, E and F are the midpoints of BC and AD, G is the midpoint of the segment EF. Construct a plane through G intersecting the segments AB, AC, AD in the points M, N, P respectively in such a way that the sum of the volumes of the tetrahedrons $BMNP$, $CMNP$ and $DMNP$ to be minimal.

H. Lesov