Round 4

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by jasperE3

– Day 1

Problem 1 Prove that there are don't exist integers a, b, c such that for every integer x the number $A = (x+a)(x+b)(x+c) - x^3 - 1$ is divisible by 9.

I. Tonov

Problem 2 Solve the system of equations:

$$\begin{cases}\sqrt{\frac{y(t-y)}{t-x}-\frac{4}{x}}+\sqrt{\frac{z(t-z)}{t-x}-\frac{4}{x}}=\sqrt{x}\\ \sqrt{\frac{z(t-z)}{t-y}-\frac{4}{y}}+\sqrt{\frac{x(t-x)}{t-y}-\frac{4}{y}}=\sqrt{y}\\ \sqrt{\frac{x(t-x)}{t-z}-\frac{4}{z}}+\sqrt{\frac{y(t-y)}{t-z}-\frac{4}{z}}=\sqrt{z}\\ x+y+z=2t\end{cases}$$

if the following conditions are satisfied: $0 < x < t$, $0 < y < t$, $0 < z < t$.

H. Lesov

Problem 3 Prove the equality:

$$\sum_{k=1}^{n-1}\frac{1}{\sin^2\frac{(2k+1)\pi}{2n}}=n^2$$

where n is a natural number.

H. Lesov

– Day 2

Problem 4 Find maximal possible number of points lying on or inside a circle with radius R in such a way that the distance between every two points is greater than $R\sqrt{2}$.

H. Lesov

Problem 5 In a circle with radius R, there is inscribed a quadrilateral with perpendicular diagonals. From the intersection point of the diagonals, there are perpendiculars drawn to the sides of the quadrilateral.

(a) Prove that the feet of these perpendiculars P_1, P_2, P_3, P_4 are vertices of the quadrilateral that is inscribed and circumscribed.

(b) Prove the inequalities $2r_1 \leq \sqrt{2}R_1 \leq R$ where R_1 and r_1 are radii respectively of the circumcircle and inscircle to the quadrilateral $P_1P_2P_3P_4$. When does equality hold?

H. Lesov

Problem 6 It is given a tetrahedron $ABCD$ for which two points of opposite edges are mutually perpendicular. Prove that:

(a) the four altitudes of $ABCD$ intersects at a common point H;
(b) $AH + BH + CH + DH < p + 2R$, where p is the sum of the lengths of all edges of $ABCD$ and R is the radii of the sphere circumscribed around $ABCD$.

H. Lesov
