Round 4

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– Day 1

Problem 1 A natural number is called *triangular* if it may be presented in the form $\frac{n(n+1)}{2}$. Find all values of a $(1\le a\le 9)$ for which there exist a triangular number all digit of which are equal to a.

Problem 2 Prove that the equation

$$\sqrt{2-x^2}+\sqrt[3]{3-x^3}=0$$

has no real solutions.

Problem 3 There are given 20 points in the plane, no three of which lie on a single line. Prove that there exist at least 969 quadrilaterals with vertices from the given points.

– Day 2

Problem 4 It is given a triangle ABC. Let R be the radius of the circumcircle of the triangle and O_1,O_2,O_3 be the centers of excircles of the triangle ABC and q is the perimeter of the triangle $O_1O_2O_3$. Prove that $q\le 6R\sqrt3$. When does equality hold?

Problem 5 Let $A_1,A_2,\ldots,A_{2n}$ are the vertices of a regular $2n$-gon and P is a point from the incircle of the polygon. If $\alpha_i=\angle A_iPA_{i+n}$, $i=1,2,\ldots,n$. Prove the equality

$$\sum_{i=1}^n\tan^2\alpha_i=2n\frac{\cos^2\frac{\pi}{2n}}{\sin^4\frac{\pi}{2n}}.$$

Problem 6 In a triangular pyramid $SABC$ one of the plane angles with vertex S is a right angle and the orthogonal projection of S on the base plane ABC coincides with the orthocenter of the triangle ABC. Let $SA=m$, $SB=n$, $SC=p$, r is the inradius of ABC. H is the height of the pyramid and r_1,r_2,r_3 are radii of the incircles of the intersections of the pyramid with the plane passing through SA,SB,SC and the height of the pyramid. Prove that

(a) $m^2+n^2+p^2\ge 18r^2$;
(b) $\frac{r_1}{H},\frac{r_2}{H},\frac{r_3}{H}$ are in the range $(0.4,0.5)$.