**Round 4**

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by jasperE3

– Day 1

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**Problem 1** Find all natural numbers $a > 1$, with the property that every prime divisor of $a^6 - 1$ divides also at least one of the numbers $a^3 - 1$, $a^2 - 1$.

*K. Dochev*

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**Problem 2** Two bicyclists traveled the distance from $A$ to $B$, which is $100$ km, with speed $30$ km/h and it is known that the first started $30$ minutes before the second. $20$ minutes after the start of the first bicyclist from $A$, there is a control car started whose speed is $90$ km/h and it is known that the car is reached the first bicyclist and is driving together with him for $10$ minutes, went back to the second and was driving for $10$ minutes with him and after that the car is started again to the first bicyclist with speed $90$ km/h and etc. to the end of the distance. How many times will the car drive together with the first bicyclist?

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**Problem 3** On a chessboard (with $64$ squares) there are situated $32$ white and $32$ black pools. We say that two pools form a mixed pair when they are with different colors and they lie on the same row or column. Find the maximum and the minimum of the mixed pairs for all possible situations of the pools.

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– Day 2

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**Problem 4** Let $\delta_0 = \triangle A_0B_0C_0$ be a triangle. On each of the sides $B_0C_0$, $C_0A_0$, $A_0B_0$, there are constructed squares in the halfplane, not containing the respective vertex $A_0, B_0, C_0$ and $A_1, B_1, C_1$ are the centers of the constructed squares. If we use the triangle $\delta_1 = \triangle A_1B_1C_1$ in the same way we may construct the triangle $\delta_2 = \triangle A_2B_2C_2$; from $\delta_2 = \triangle A_2B_2C_2$ we may construct $\delta_3 = \triangle A_3B_3C_3$ and etc. Prove that:

(a) segments $A_0A_1, B_0B_1, C_0C_1$ are respectively equal and perpendicular to $B_1C_1, C_1A_1, A_1B_1$;
(b) vertices $A_1, B_1, C_1$ of the triangle $\delta_1$ lies respectively over the segments $A_0A_3, B_0B_3, C_0C_3$ (defined by the vertices of $\delta_0$ and $\delta_1$) and divide them in ratio $2 : 1$.

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**Problem 5** Prove that for $n \geq 5$ the side of regular inscribable $n$-gon is bigger than the side of regular $n + 1$-gon circumscribed around the same circle and if $n \leq 4$ the opposite statement is true.

**Problem 6** In space, we are given the points $A, B, C$ and a sphere with center $O$ and radius $1$. Find the point $X$ from the sphere for which the sum $f(X) = |XA|^2 + |XB|^2 + |XC|^2$ attains its maximal and minimal value. Prove that if the segments $OA, OB, OC$ are pairwise perpendicular and $d$ is the distance from the center $O$ to the centroid of the triangle $ABC$ then:

(a) the maximum of $f(X)$ is equal to $9d^2 + 3 + 6d$;
(b) the minimum of $f(X)$ is equal to $9d^2 + 3 - 6d$.

*K. Dochev and I. Dimovski*