Round 4

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by jasperE3

– Day 1

Problem 1 Prove that if the sum of x^5, y^5 and z^5, where x, y and z are integer numbers, is divisible by 25 then the sum of some two of them is divisible by 25.

Problem 2 Prove that

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} < 2$$

for every $n \in \mathbb{N}$.

Problem 3 Some of the points in the plane are white and some are blue (every point of the plane is either white or blue). Prove that for every positive number r:

(a) there are at least two points with different color such that the distance between them is equal to r;
(b) there are at least two points with the same color and the distance between them is equal to r;
(c) will the statements above be true if the plane is replaced with the real line?

– Day 2

Problem 4 Find the sides of a triangle if it is known that the inscribed circle meets one of its medians in two points and these points divide the median into three equal segments and the area of the triangle is equal to $6\sqrt{14}$ cm^2.

Problem 5 Prove the equality

$$\prod_{k=1}^{2m} \cos \frac{k\pi}{2m+1} = \frac{(-1)^m}{4m}.$$

Problem 6 It is given that $r = \left(3\left(\sqrt{6}-1\right) - 4\left(\sqrt{3}+1\right) + 5\sqrt{2}\right) R$ where r and R are the radii of the inscribed and circumscribed spheres in a regular n-angled pyramid. If it is known that the centers of the spheres given coincide,

(a) find n;
(b) if $n = 3$ and the lengths of all edges are equal to a find the volumes of the parts from the pyramid after drawing a plane μ, which intersects two of the edges passing through point A respectively in the points E and F in such a way that $|AE| = p$ and $|AF| = q$ $(p < a, q < a)$,

intersects the extension of the third edge behind opposite of the vertex A wall in the point G in such a way that $|AG| = t$ $(t > a)$.