Round 4

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by jasperE3

– Day 1

Problem 1 Find all natural values of k for which the system

$$\begin{cases} x_1+x_2+\ldots+x_k=9 \\ \frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_k}=1 \end{cases}$$

has solutions in positive numbers. Find these solutions.

I. Dimovski

Problem 2 Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the inequality

$$xf(y)+yf(x)=(x+y)f(x)f(y)$$

for all reals x,y. Prove that exactly two of them are continuous.

I. Dimovski

Problem 3 Prove that a binomial coefficient $\binom{n}{k}$ is odd if and only if all digits 1 of k, when k is written in binary, are on the same positions when n is written in binary.

I. Dimovski

– Day 2

Problem 4 On the line g we are given the segment AB and a point C not on AB. Prove that on g, there exists at least one pair of points P,Q symmetrical with respect to C, which divide the segment AB internally and externally in the same ratios, i.e

$$\frac{PA}{PB}=\frac{QA}{QB} \qquad (1)$$

If A,B,P,Q are such points from the line g satisfying (1), prove that the midpoint C of the segment PQ is the external point for the segment AB.

K. Petrov

Problem 5 The point M is inside the tetrahedron $ABCD$ and the intersection points of the lines AM,BM,CM and DM with the opposite walls are denoted with A_1,B_1,C_1,D_1 respectively. It is given also that the ratios $\frac{MA}{MA_1}$, $\frac{MB}{MB_1}$, $\frac{MC}{MC_1}$, and $\frac{MD}{MD_1}$ are equal to the same number k. Find all possible values of k.

K. Petrov

Problem 6 Find the kind of a triangle if

$$\frac{a\cos\alpha+b\cos\beta+c\cos\gamma}{a\sin\alpha+b\sin\beta+c\sin\gamma}=\frac{2p}{9R}.$$

(α,β,γ are the measures of the angles, a,b,c are the respective lengths of the sides, p the semiperimeter, R is the circumradius)

K. Petrov
