Round 4

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Problem 1 Prove that the equation

$$3x(x-3y)=y^2+z^2$$

doesn't have any integer solutions except $x=0, y=0, z=0$.

Problem 2 Prove that for every four positive numbers a, b, c, d the following inequality is true:

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt[3]{\frac{abc+abd+acd+bcd}{4}}.$$

Problem 3 (a) In the plane of the triangle ABC, find a point with the following property: its symmetrical points with respect to the midpoints of the sides of the triangle lie on the circumscribed circle.
(b) Construct the triangle ABC if it is known the positions of the orthocenter H, midpoint of the side AB and the midpoint of the segment joining the feet of the heights through vertices A and B.

Problem 4 It is given a tetrahedron with vertices A, B, C, D.

(a) Prove that there exists a vertex of the tetrahedron with the following property: the three edges of that tetrahedron through that vertex can form a triangle.
(b) On the edges DA, DB and DC there are given the points M, N and P for which:

$$DM=\frac{DA}{n}, \quad DN=\frac{DB}{n+1} \quad DP=\frac{DC}{n+2}$$

where n is a natural number. The plane defined by the points M, N and P is α_n. Prove that all planes α_n, $(n=1,2,3,\ldots)$ pass through a single straight line.
