**Round 4**

www.artofproblemsolving.com/community/c2384811

by jasperE3

**Problem 1** The numbers $2, 3, 7$ have the property that the product of any two of them increased by $1$ is divisible by the third number. Prove that this triple of integer numbers greater than $1$ is the only triple with the given property.

---

**Problem 2** Prove the inequality:

$$(1+\sin^2\alpha)^n + (1+\cos^2\alpha)^n \geq 2\left(\frac{3}{2}\right)^n$$

is true for every natural number $n$. When does equality hold?

---

**Problem 3** In the triangle $ABC$, angle bisector $CD$ intersects the circumcircle of $ABC$ at the point $K$.

(a) Prove the equalities:

$$\frac{1}{ID} - \frac{1}{IK} = \frac{1}{CI}, \quad \frac{CI}{ID} - \frac{ID}{DK} = 1$$

where $I$ is the center of the inscribed circle of triangle $ABC$.

(b) On the segment $CK$ some point $P$ is chosen whose projections on $AC, BC, AB$ respectively are $P_1, P_2, P_3$. The lines $PP_3$ and $P_1P_2$ intersect at a point $M$. Find the locus of $M$ when $P$ moves around segment $CK$.

---

**Problem 4** In the space there are given crossed lines $s$ and $t$ such that $\angle(s,t) = 60^\circ$ and a segment $AB$ perpendicular to them. On $AB$ it is chosen a point $C$ for which $AC : CB = 2 : 1$ and the points $M$ and $N$ are moving on the lines $s$ and $t$ in such a way that $AM = 2BN$. The angle between vectors $\overrightarrow{AM}$ and $\overrightarrow{BM}$ is $60^\circ$. Prove that:

(a) the segment $MN$ is perpendicular to $t$;

(b) the plane $\alpha$, perpendicular to $AB$ in point $C$, intersects the plane $CMN$ on fixed line $\ell$ with given direction in respect to $s$;

(c) all planes passing by $ell$ and perpendicular to $AB$ intersect the lines $s$ and $t$ respectively at points $M$ and $N$ for which $AM = 2BN$ and $MN \perp t$.

---


© 2021 AoPS Incorporated

Art of Problem Solving is an ACS WASC Accredited School.