Round 4

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Problem 1 Find all three-digit numbers whose remainders after division by 11 give quotient, equal to the sum of the squares of its digits.

Problem 2 It is given the equation $x^2 + px + 1 = 0$, with roots x_1 and x_2;

(a) find a second-degree equation with roots y_1, y_2 satisfying the conditions $y_1 = x_1(1 - x_1)$, $y_2 = x_2(1 - x_2)$;
(b) find all possible values of the real parameter p such that the roots of the new equation lies between -2 and 1.

Problem 3 In the trapezium $ABCD$, a point M is chosen on the non-base segment AB. Through the points M, A, D and M, B, C are drawn circles k_1 and k_2 with centers O_1 and O_2. Prove that:

(a) the second intersection point N of k_1 and k_2 lies on the other non-base segment CD or on its continuation;
(b) the length of the line O_1O_2 doesnt depend on the location of M on AB;
(c) the triangles O_1MO_2 and DMC are similar. Find such a position of M on AB that makes k_1 and k_2 have the same radius.

Problem 4 In the tetrahedron $ABCD$ three of the faces are right-angled triangles and the other is not an obtuse triangle. Prove that:

(a) the fourth wall of the tetrahedron is a right-angled triangle if and only if exactly two of the plane angles having common vertex with the some of vertices of the tetrahedron are equal.
(b) its volume is equal to $\frac{1}{6}$ multiplied by the multiple of two shortest edges and an edge not lying on the same wall.