German National Olympiad 2021, Final Round

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–	Day 1
1	Determine all real numbers a, b, c and d with the following property: The numbers a and b are distinct roots of $2x^2 - 3cx + 8d$ and the numbers c and d are distinct roots of $2x^2 - 3ax + 8b$.
2	Let P on AB, Q on BC, R on CD and S on AD be points on the sides of a convex quadrilateral $ABCD$. Show that the following are equivalent: (1) There is a choice of P, Q, R, S, for which all of them are interior points of their side, such that $PQRS$ has minimal perimeter. (2) $ABCD$ is a cyclic quadrilateral with circumcenter in its interior.
3	For a fixed k with $4 \leq k \leq 9$ consider the set of all positive integers with k decimal digits such that each of the digits from 1 to k occurs exactly once. Show that it is possible to partition this set into two disjoint subsets such that the sum of the cubes of the numbers in the first set is equal to the sum of the cubes in the second set.
–	Day 2
4	Let OFT and NOT be two similar triangles (with the same orientation) and let $FANO$ be a parallelogram. Show that $$\lvert OF\rvert \cdot \lvert ON\rvert = \lvert OA\rvert \cdot \lvert OT\rvert.$$
5	a) Determine the largest real number A with the following property: For all non-negative real numbers x, y, z, one has $$\frac{1+yz}{1+x^2} + \frac{1+zx}{1+y^2} + \frac{1+xy}{1+z^2} \geq A.$$ b) For this real number A, find all triples (x, y, z) of non-negative real numbers for which equality holds in the above inequality.
6	Determine whether there are infinitely many triples (u, v, w) of positive integers such that u, v, w form an arithmetic progression and the numbers $uv + 1, vw + 1$ and $wu + 1$ are all perfect squares.