**ELMO Problems 2021**

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**–** Day 1

**1** In $\triangle ABC$, points $P$ and $Q$ lie on sides $AB$ and $AC$, respectively, such that the circumcircle of $\triangle APQ$ is tangent to $BC$ at $D$. Let $E$ lie on side $BC$ such that $BD = EC$. Line $DP$ intersects the circumcircle of $\triangle CDQ$ again at $X$, and line $DQ$ intersects the circumcircle of $\triangle BDP$ again at $Y$. Prove that $D$, $E$, $X$, and $Y$ are concyclic.

**2** Let $n > 1$ be an integer and let $a_1, a_2, \dots, a_n$ be integers such that $n \mid a_i - i$ for all integers $1 \le i \le n$. Prove there exists an infinite sequence $b_1, b_2, \dots$ such that

- $b_k \in \{a_1, a_2, \dots, a_n\}$ for all positive integers $k$, and
- $\sum\limits_{k=1}^{\infty} \frac{b_k}{n^k}$ is an integer.

**3** Each cell of a $100 \times 100$ grid is colored with one of $101$ colors. A cell is *diverse* if, among the $199$ cells in its row or column, every color appears at least once. Determine the maximum possible number of diverse cells.

**–** Day 2

**4** The set of positive integers is partitioned into $n$ disjoint infinite arithmetic progressions $S_1, S_2, \dots, S_n$ with common differences $d_1, d_2, \dots, d_n$, respectively. Prove that there exists exactly one index $1 \le i \le n$ such that

$$\frac{1}{d_i} \prod_{j=1}^{n} d_j \in S_i.$$

**5** Let $n$ and $k$ be positive integers. Two infinite sequences $\{s_i\}_{i\ge1}$ and $\{t_i\}_{i\ge1}$ are *equivalent* if, for all positive integers $i$ and $j$, $s_i = s_j$ if and only if $t_i = t_j$. A sequence $\{r_i\}_{i\ge1}$ has *equi-period* $k$ if $r_1, r_2, \dots$ and $r_{k+1}, r_{k+2}, \dots$ are equivalent.

Suppose $M$ infinite sequences with equi-period $k$ whose terms are in the set $\{1, \dots, n\}$ can be chosen such that no two chosen sequences are equivalent to each other. Determine the largest possible value of $M$ in terms of $n$ and $k$.

**6** In $\triangle ABC$, points $D$, $E$, and $F$ lie on sides $BC$, $CA$, and $AB$, respectively, such that each of the quadrilaterals $AFDE$, $BDEF$, and $CEFD$ has an incircle. Prove that the inradius of $\triangle ABC$ is twice the inradius of $\triangle DEF$.