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– Grade 1

Problem 1 Let n be a natural number. Solve the equation

$$||\cdots|||x-1|-2|-3|-\ldots-(n-1)|-n|=0.$$

Problem 2 Given are real numbers $a<b<c<d$. Determine all permutations p,q,r,s of the numbers a,b,c,d for which the value of the sum

$$(p-q)^2+(q-r)^2+(r-s)^2+(s-p)^2$$

is minimal.

Problem 3 A chord divides the interior of a circle k into two parts. Variable circles k_1 and k_2 are inscribed in these two parts, touching the chord at the same point. Show that the ratio of the radii of circles k_1 and k_2 is constant, i.e. independent of the tangency point with the chord.

Problem 4 An infinite sheet of paper is divided into equal squares, some of which are colored red. In each 2×3 rectangle, there are exactly two red squares. Now consider an arbitrary 9×11 rectangle. How many red squares does it contain? (The sides of all considered rectangles go along the grid lines.)

– Grade 2

Problem 1 In a regular hexagon $ABCDEF$ with center O, points M and N are the midpoints of the sides CD and DE, and L the intersection point of AM and BN. Prove that:

(a) ABL and $DMLN$ have equal areas;
(b) $\angle ALD=\angle OLN=60^\circ$;
(c) $\angle OLD=90^\circ$.

Problem 2 For any different positive numbers a,b,c prove the inequality

$$a^ab^bc^c>a^bb^cc^a.$$

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Problem 3 The number 2^{1997} has m decimal digits, while the number 5^{1997} has n digits. Evaluate $m+n$.

Problem 4 In the plane are given 1997 points. Show that among the pairwise distances between these points, there are at least 32 different values.

– Grade 3

Problem 1 Integers x, y, z and a, b, c satisfy

$$x^2+y^2=a^2, \;\; y^2+z^2=b^2 \;\; z^2+x^2=c^2.$$

Prove that the product xyz is divisible by (a) 5, and (b) 55.

Problem 2 Prove that for every real number x and positive integer n

$$|\cos x|+|\cos 2x|+|\cos 2^2x|+\ldots+|\cos 2^nx| \geq \frac{n}{2\sqrt{2}}.$$

Problem 3 The areas of the faces ABD, ACD, BCD, BCA of a tetrahedron $ABCD$ are S_1, S_2, Q_1, Q_2, respectively. The angle between the faces ABD and ACD equals α, and the angle between BCD and BCA is β. Prove that

$$S_1^2+S_2^2-2S_1S_2\cos\alpha = Q_1^2+Q_2^2-2Q_1Q_2\cos\beta.$$

Problem 4 On the sides of a triangle ABC are constructed similar triangles ABD, BCE, CAF with $k = AD/DB = BE/EC = CF/FA$ and $\alpha = \angle ADB = \angle BEC = \angle CFA$. Prove that the midpoints of the segments AC, BC, CD and EF form a parallelogram with an angle α and two sides whose ratio is k.

– Grade 4

Problem 1 Find the last four digits of each of the numbers 3^{1000} and 3^{1997}.

Problem 2 Consider a circle k and a point K in the plane. For any two distinct points P and Q on k, denote by k' the circle through P, Q and K. The tangent to k' at K meets the line PQ at point M. Describe the locus of the points M when P and Q assume all possible positions.

Problem 3 Function f is defined on the positive integers by $f(1)=1$, $f(2)=2$ and

$$f(n+2)=f(n+2-f(n+1))+f(n+1-f(n)) \;\text{ for } n\geq 1.$$

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(a) Prove that $f(n+1) - f(n) \in \{0, 1\}$ for each $n \geq 1$.
(b) Show that if $f(n)$ is odd then $f(n+1) = f(n) + 1$.
(c) For each positive integer k find all n for which $f(n) = 2^{k-1} + 1$.

Problem 4 Let k be a natural number. Determine the number of non-congruent triangles with the vertices at vertices of a given regular $6k$-gon.
