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by HamstPan38825

Problem 1 Given two vectors $\overrightarrow{a}$, $\overrightarrow{b}$, find the range of possible values of $\|\overrightarrow{a} - 2\overrightarrow{b}\|$ where $\|\overrightarrow{v}\|$ denotes the magnitude of a vector $\overrightarrow{v}$.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 1)

Problem 2 Compute the value of

$$\sin^2 20^\circ + \cos^2 50^\circ + \sin 20^\circ \cos 50^\circ.$$

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 2)

Problem 3 There exists complex numbers $z = x + yi$ such that the point (x, y) lies on the ellipse with equation $\frac{x^2}{9} + \frac{y^2}{16} = 1$. If $\frac{z-1-i}{z-i}$ is real, compute z.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 3)

Problem 4 When the expression

$$(xy - 5x + 3y - 15)^n$$

for some positive integer n is expanded and like terms are combined, the expansion contains at least 2021 distinct terms. Compute the minimum possible value of n.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 4)

Problem 5 Define the regions M, N in the Cartesian Plane as follows:

$$M = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le \text{min}(2x, 3 - x)\}$$
$$N = \{(x, y) \in \mathbb{R}^2 \mid t \le x \le t + 2\}$$

for some real number t. Denote the common area of M and N for some t be $f(t)$. Compute the algebraic form of the function $f(t)$ for $0 \le t \le 1$.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 5)

Problem 6 A sequence $\{a_n\}$ satisfies

$$a_0 = 0, a_1 = a_2 = 1, a_{3n} = a_n, a_{3n+1} = a_{3n+2} = a_n + 1$$

for all $n \ge 1$. Compute a_{2021}.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 6)

Problem 7 For two sets A, B, define the operation

$$A \otimes B = \{x \mid x = ab + a + b, a \in A, b \in B\}.$$

Set $A = \{0, 2, 4, \cdots, 18\}$ and $B = \{98, 99, 100\}$. Compute the sum of all the elements in $A \otimes B$.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 7)

Problem 8 In $\triangle ABC$, $\angle B = \angle C = 30^\circ$ and $BC = 2\sqrt{3}$. P, Q lie on segments $\overline{AB}, \overline{AC}$ such that $AP = 1$ and $AQ = \sqrt{2}$. Let D be the foot of the altitude from A to BC. We fold $\triangle ABC$ along line AD in three dimensions such that the dihedral angle between planes ADB and ADC equals 60 degrees. Under this transformation, compute the length PQ.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 8)

Problem 9 Let $\triangle ABC$ have its vertices at $A(0,0), B(7,0), C(3,4)$ in the Cartesian plane. Construct a line through the point $(6-2\sqrt{2}, 3-\sqrt{2})$ that intersects segments AC, BC at P, Q respectively. If $[PQC] = \frac{14}{3}$, what is $|CP| + |CQ|$?

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 9)

Problem 10 Define the sequence a_n by the rule

$$a_{n+1} = \left\lfloor \frac{a_n}{2} \right\rfloor + \left\lfloor \frac{a_n}{3} \right\rfloor$$

for $n \in \{1, 2, 3, 4, 5, 6, 7\}$, where $\lfloor x \rfloor$ denotes the greatest integer not greater than x. If $a_8 = 8$, how many possible values are there for a_1 given that it is a positive integer?

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 10)

Problem 11 The function $f(x) = x^2 + ax + b$ has two distinct zeros. If $f(x^2 + 2x - 1) = 0$ has four distinct zeros $x_1 < x_2 < x_3 < x_4$ that form an arithmetic sequence, compute the range of $a - b$.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 11)

Problem 12 Let C be the left vertex of the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ in the Cartesian Plane. For some real number k, the line $y = kx + 1$ meets the ellipse at two distinct points A, B.

(i) Compute the maximum of $|CA| + |CB|$.
(ii) Let the line $y = kx + 1$ meet the x and y axes at M and N, respectively. If the intersection of the perpendicular bisector of MN and the circle with diameter MN lies inside the given ellipse, compute the range of possible values of k.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 12)

Problem 13 Let n be a given positive integer. The sequence of real numbers $a_1, a_2, a_3, \cdots, a_n$ satisfy for each $m \le n$,

$$\left|\sum_{k=1}^{m} \frac{a_k}{k}\right| \le 1.$$

Given this information, find the greatest possible value of $|\sum_{k=1}^{n} a_k|$.

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 13)

Problem 14 Define the set $P = \{a_1, a_2, a_3, \cdots, a_n\}$ and its arithmetic mean

$$C_p = \frac{a_1 + a_2 + \cdots + a_m}{m}.$$

If we divide $S = \{1, 2, 3, \cdots, n\}$ into two disjoint subsets A, B, compute the greatest possible value of $|C_A - C_B|$. For how many (A, B) is equality attained?

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 14)

Problem 15 Positive real numbers x, y, z satisfy $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that

$$\frac{x^4 + y^2 z^2}{x^{\frac{5}{2}}(y+z)} + \frac{y^4 + z^2 x^2}{y^{\frac{5}{2}}(z+x)} + \frac{z^4 + y^2 x^2}{z^{\frac{5}{2}}(y+x)} \ge 1.$$

(Source: China National High School Mathematics League 2021, Zhejiang Province, Problem 15)
