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– day 1

1 Let $n \geq 2$ be a fixed positive integer. Let $\{a_1, a_2, ..., a_n\}$ be fixed positive integers whose sum is $2n-1$. Denote by $S_{\mathbb{A}}$ the sum of elements of a set A. Find the minimal and maximal value of $S_{\mathbb{X}} \cdot S_{\mathbb{Y}}$ where $\mathbb{X}$ and $\mathbb{Y}$ are two sets with the property that $\mathbb{X} \cup \mathbb{Y} = \{a_1, a_2, ..., a_n\}$ and $\mathbb{X} \cap \mathbb{Y} = \emptyset$.

[i]Note: $\mathbb{X}$ and $\mathbb{Y}$ can have multiple equal elements. For example, when $n = 5$ and $a_1 = ... = a_4 = 1$ and $a_5 = 5$, we can consider $\mathbb{X} = \{1, 1, 1\}$ and $\mathbb{Y} = \{1, 5\}$. Moreover, in this case, $S_{\mathbb{X}} = 3$ and $S_{\mathbb{Y}} = 6$.[/i]

2 Angle bisector at A, altitude from B to CA and altitude of C to AB on a scalene triangle ABC forms a triangle $\triangle$. Let P and Q points on lines AB and AC, respectively, such that the midpoint of segment PQ is the orthocenter of the triangle $\triangle$. Prove that the points B, C, P and Q lie on a circle.

3 Let $\mathcal{S}$ be a set consisting of $n \geq 3$ positive integers, none of which is a sum of two other distinct members of $\mathcal{S}$. Prove that the elements of $\mathcal{S}$ may be ordered as $a_1, a_2, \dots, a_n$ so that a_i does not divide $a_{i-1} + a_{i+1}$ for all $i = 2, 3, \dots, n-1$.

– day 2

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y satisfies,

$$2 + f(x)f(y) \leq xy + 2f(x+y+1).$$

5 Find all positive integers n such that the number $n^5 + 79$ has all the same digits when it is written in decimal represantation.

6 Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \dots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote $r(n)$ the fraction of integers $\{1, 2, \dots, n\}$ that are red.
Prove that there exists a positive integer $a \in \{1, 2, \dots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \dots, N$.

Netherlands