South East Mathematical Olympiad 2021

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–	Grade 10
–	Day 1
1	A sequence $\{a_n\}$ is defined recursively by $a_1=\frac{1}{2}$, and for $n\geq 2$, $0<a_n\leq a_{n-1}$ and $$a_n^2(a_{n-1}+1)+a_{n-1}^2(a_n+1)-2a_na_{n-1}(a_na_{n-1}+a_n+1)=0.$$ (1) Determine the general formula of the sequence $\{a_n\}$; (2) Let $S_n=a_1+\cdots+a_n$. Prove that for $n\geq 1$, $\ln\left(\frac{n}{2}+1\right)<S_n<\ln(n+1)$.
2	In $\triangle ABC$,$AB=AC>BC$, point O,H are the circumcenter and orthocenter of $\triangle ABC$ respectively,G is the midpoint of segment AH , BE is the altitude on AC . Prove that if $OE \parallel BC$, then H is the incenter of $\triangle GBC$.
3	Let p be an odd prime and $\{u_i\}_{i\ge 0}$be an integer sequence. Let $v_n=\sum_{i=0}^{n} C_{n}^{i} p^iu_i$ where C_n^i denotes the binomial coefficients. If $v_n=0$ holds for infinitely many n , prove that it holds for every positive integer n.
4	Suppose there are $n\geq 5$ different points arbitrarily arranged on a circle, the labels are $1, 2,\dots$, and n, and the permutation is S. For a permutation , a “descending chain” refers to several consecutive points on the circle , and its labels is a clockwise descending sequence (the length of sequence is at least 2), and the descending chain cannot be extended to longer .The point with the largest label in the chain is called the ”starting point of descent”, and the other points in the chain are called the “non-starting point of descent” . For example: there are two descending chains $5, 2$and $4, 1$ in $5, 2, 4, 1, 3$ arranged in a clockwise direction, and 5 and 4 are their starting points of descent respectively, and $2, 1$ is the non-starting point of descent . Consider the following operations: in the first round, find all descending chains in the permutation S, delete all non-starting points of descent , and then repeat the first round of operations for the arrangement of the remaining points, until no more descending chains can be found. Let $G(S)$ be the number of all descending chains that permutation S has appeared in the operations, $A(S)$ be the average value of $G(S)$of all possible n-point permutations S. (1) Find $A(5)$. (2)For $n\geq 6$, prove that $\frac{83}{120}n-\frac{1}{2} \leq A(S) \leq \frac{101}{120}n-\frac{1}{2}$.
–	Day 2

5 To commemorate the $43rd$ anniversary of the restoration of mathematics competitions, a mathematics enthusiast
arranges the first 2021 integers $1, 2, \dots, 2021$ into a sequence $\{a_n\}$ in a certain order, so that the sum of any consecutive 43 items in the sequence is a multiple of 43.
(1) If the sequence of numbers is connected end to end into a circle, prove that the sum of any consecutive 43 items on the circle is also a multiple of 43;
(2) Determine the number of sequences $\{a_n\}$ that meets the conditions of the question.

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6 Let $ABCD$ be a cyclic quadrilateral. Let E be a point on side BC, F be a point on side AE, G be a point on the exterior angle bisector of $\angle BCD$, such that $EG = FG$, $\angle EAG = \frac{1}{2}\angle BAD$. Prove that $AB \cdot AF = AD \cdot AE$.

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7 Let a, b, c be pairwise distinct positive real, Prove that

$$\frac{ab+bc+ca}{(a+b)(b+c)(c+a)} < \frac{1}{7}(\frac{1}{|a-b|} + \frac{1}{|b-c|} + \frac{1}{|c-a|}).$$

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8 Determine all the pairs of positive integers (a, b), such that

$$14\varphi^2(a) - \varphi(ab) + 22\varphi^2(b) = a^2 + b^2,$$

where $\varphi(n)$ is Euler's totient function.

– Grade 11

– Day 1

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1 Same as Grade 10 P2

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2 Let $p \geq 5$ be a prime number, and set $M = \{1, 2, \cdots, p-1\}$. Define

$$T = \{(n, x_n) : p | nx_n - 1 \text{ and } n, x_n \in M\}.$$

If $\sum_{(n,x_n)\in T} n \left[\frac{nx_n}{p}\right] \equiv k \pmod{p}$, with $0 \leq k \leq p-1$, where $[\alpha]$ denotes the largest integer that does not exceed α, determine the value of k.

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3 Let $a, b, c \geq 0$ and $a^2 + b^2 + c^2 \leq 1$. Prove that

$$\frac{a}{a^2+bc+1} + \frac{b}{b^2+ca+1} + \frac{c}{c^2+ab+1} + 3abc < \sqrt{3}$$

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4 For positive integer k, we say that it is a *Taurus integer* if we can delete one element from the set $M_k = \{1, 2, \cdots, k\}$, such that the sum of remaining $k-1$ elements is a positive perfect square. For example, 7 is a Taurus integer, because if we delete 3 from $M_7 = \{1, 2, 3, 4, 5, 6, 7\}$, the sum of remaining 6 elements is 25, which is a positive perfect square. (1) Determine whether 2021 is a Taurus integer. (2) For positive integer n, determine the number of Taurus integers in $\{1, 2, \cdots, n\}$.

– Day 2

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5 Let $A = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ be a set with $2n$ distinct elements, and $B_i \subseteq A$ for any $i = 1, 2, \cdots, m$. If $\bigcup_{i=1}^{m} B_i = A$, we say that the ordered $m-$tuple $(B_1, B_2, \cdots, B_m)$ is an ordered $m-$covering of A. If $(B_1, B_2, \cdots, B_m)$ is an ordered $m-$covering of A, and for any $i = 1, 2, \cdots, m$ and any $j = 1, 2, \cdots, n$, $\{a_j, b_j\}$ is not a subset of B_i, then we say that ordered $m-$tuple $(B_1, B_2, \cdots, B_m)$ is an ordered $m-$covering of A without pairs. Define $a(m, n)$ as the number of the ordered $m-$coverings of A, and $b(m, n)$ as the number of the ordered $m-$coverings of A without pairs. (1) Calculate $a(m, n)$ and $b(m, n)$. (2) Let $m \geq 2$, and there is at least one positive integer n, such that $\dfrac{a(m, n)}{b(m, n)} \leq 2021$, Determine the greatest possible values of m.

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6 Let $ABCD$ be a cyclic quadrilateral. The internal angle bisector of $\angle BAD$ and line BC intersect at E. M is the midpoint of segment AE. The exterior angle bisector of $\angle BCD$ and line AD intersect at F. The lines MF and AB intersect at G. Prove that if $AB = 2AD$, then $MF = 2MG$.

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7 Determine all the pairs of positive odd integers (a, b), such that $a, b > 1$ and

$$7\varphi^2(a) - \varphi(ab) + 11\varphi^2(b) = 2(a^2 + b^2),$$

where $\varphi(n)$ is Euler's totient function.

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8 A sequence $\{z_n\}$ satisfies that for any positive integer i, $z_i \in \{0, 1, \cdots, 9\}$ and $z_i \equiv i - 1 \pmod{10}$. Suppose there is 2021 non-negative reals $x_1, x_2, \cdots, x_{2021}$ such that for $k = 1, 2, \cdots, 2021$,

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} z_i, \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} z_i + \sum_{j=1}^{10} \frac{10 - j}{50} z_{k+j}.$$

Determine the least possible value of $\sum_{i=1}^{2021} x_i^2$.
