Tuymaada Olympiad 2021

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–	Juniors
–	Day 1
1	Quadratic trinomials F and G satisfy $F(F(x)) > F(G(x)) > G(G(x))$ for all real x. Prove that $F(x) > G(x)$ for all real x.
2	The bisector of angle B of a parallelogram $ABCD$ meets its diagonal AC at E, and the external bisector of angle B meets line AD at F. M is the midpoint of BE. Prove that $CM//EF$.
3	For n distinct positive integers all their $n(n-1)/2$ pairwise sums are considered. For each of these sums Ivan has written on the board the number of original integers which are less than that sum and divide it. What is the maximum possible sum of the numbers written by Ivan?
4	Some manors of Lipshire county are connected by roads. The inhabitants of manors connected by a road are called neighbours. Is it always possible to settle in each manor a knight (who always tells truth) or a liar (who always lies) so that every inhabitant can say ”The number of liars among my neighbours is at least twice the number of knights”?
–	Day 2
5	In a 100×100 table 110 unit squares are marked. Is it always possible to rearrange rows and columns so that all the marked unit squares are above the main diagonal or on it?
6	Given are real $y > 1$ and positive integer $n \leq y^{50}$ such that all prime divisors of n do not exceed y. Prove that n is a product of 99 positive integer factors (not necessarily primes) not exceeding y.
7	A pile contains 2021^{2021} stones. In a move any pile can be divided into two piles so that the numbers of stones in them differ by a power of 2 with non-negative integer exponent. After some move it turned out that the number of stones in each pile is a power of 2 with non-negative integer exponent. Prove that the number of moves performed was even.

8 An acute triangle ABC is given, $AC \neq BC$. The altitudes drawn from A and B meet at H and intersect the external bisector of the angle C at Y and X respectively. The external bisector of the angle AHB meets the segments AX and BY at P and Q respectively. If $PX = QY$, prove that $AP + BQ \geq 2CH$.

– Seniors

– Day 1

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1 Polynomials F and G satisfy:

$$F(F(x)) > G(F(x)) > G(G(x))$$

for all real x.Prove that $F(x) > G(x)$ for all real x.

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2 In trapezoid $ABCD$,M is the midpoint of the base AD.Point E lies on the segment BM.It is known that $\angle ADB = \angle MAE = \angle BMC$.Prove that the triangle BCE is isosceles.

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3 Positive real numbers $a_1, \dots, a_k, b_1, \dots, b_k$ are given. Let $A = \sum_{i=1}^k a_i, B = \sum_{i=1}^k b_i$. Prove the inequality

$$\left(\sum_{i=1}^k \frac{a_i b_i}{a_i B + b_i A} - 1\right)^2 \geq \sum_{i=1}^k \frac{a_i^2}{a_i B + b_i A} \cdot \sum_{i=1}^k \frac{b_i^2}{a_i B + b_i A}.$$

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4 An $n \times n$ square (n is a positive integer) consists of n^2 unit squares.A *monotonous path* in this square is a path of length $2n$ beginning in the left lower corner of the square,ending in its right upper corner and going along the sides of unit squares.
For each k, $0 \leq k \leq 2n-1$, let S_k be the set of all the monotonous paths such that the number of unit squares lying below the path leaves remainder k upon division by $2n-1$.Prove that all S_k contain equal number of elements.

– Day 2

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6 In a $n \times n$ table $(n > 1)$ k unit squares are marked.One wants to rearrange rows and columns so that all the marked unit squares are above the main diagonal or on it.For what maximum k is it always possible?

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5 Sines of three acute angles form an arithmetic progression, while the cosines of these angles form a geometric progression. Prove that all three angles are equal.

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7 same as Juniors P8

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8 In a sequence P_n of quadratic trinomials each trinomial, starting with the third, is the sum of the two preceding trinomials. The first two trinomials do not have common roots. Is it possible that P_n has an integral root for each n?