Bulgaria National Olympiad 1993, Round 4

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– Day 1

1 Find all functions f , defined and having values in the set of integer numbers, for which the following conditions are satisfied:
(a) $f(1)=1$;
(b) for every two whole (integer) numbers m and n, the following equality is satisfied:

$$f(m+n)(f(m)-f(n))=f(m-n)(f(m)+f(n))$$

2 Let M be an interior point of the triangle ABC such that $AMC = 90^\circ$, $AMB = 150^\circ$, and $BMC = 120^\circ$. The circumcenters of the triangles AMC, AMB, and BMC are P, Q, and R respectively. Prove that the area of ΔPQR is greater than or equal to the area of ΔABC.

3 it is given a polyhedral constructed from two regular pyramids with bases heptagons (a polygon with 7 vertices) with common base $A_1A_2A_3A_4A_5A_6A_7$ and vertices respectively the points B and C. The edges BA_i, CA_i $(i=1,...,7)$, diagonals of the common base are painted in blue or red. Prove that there exists three vertices of the polyhedral given which forms a triangle with all sizes in the same color.

– Day 2

4 Find all natural numbers $n > 1$ for which there exists such natural numbers $a_1, a_2, ..., a_n$ for which the numbers $\{a_i + a_j | 1 \le i \le j \le n\}$ form a full system modulo $\frac{n(n+1)}{2}$.

5 Let Oxy be a fixed rectangular coordinate system in the plane.
Each ordered pair of points A_1, A_2 from the same plane which are different from O and have coordinates x_1, y_1 and x_2, y_2 respectively is associated with real number $f(A_1, A_2)$ in such a way that the following conditions are satisfied:

(a) If $OA_1 = OB_1$, $OA_2 = OB_2$ and $A_1A_2 = B_1B_2$ then $f(A_1, A_2) = f(B_1, B_2)$.

(b) There exists a polynomial of second degree $F(u, v, w, z)$ such that $f(A_1, A_2) = F(x_1, y_1, x_2, y_2)$.

(c) There exists such a number $\phi \in (0, \pi)$ that for every two points A_1, A_2 for which $\angle A_1OA_2 = \phi$ is satisfied $f(A_1, A_2) = 0$.

(d) If the points A_1, A_2 are such that the triangle OA_1A_2 is equilateral with side 1 then $f(A_1, A_2) = \frac{1}{2}$.

Prove that $f(A_1, A_2) = \overrightarrow{OA_1} \cdot \overrightarrow{OA_2}$ for each ordered pair of points A_1, A_2.

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6 Find all natural numbers n for which there exists set S consisting of n points in the plane, satisfying the condition:
For each point $A \in S$ there exist at least three points say X, Y, Z from S such that the segments AX, AY andAZ have length 1 (it means that $AX = AY = AZ = 1$).
