Bulgaria National Olympiad 1994, Round 4

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–	Day 1
1	Two circles $k_1(O_1, R)$ and $k_2(O_2, r)$ are given in the plane such that $R \geq \sqrt{2}r$ and $$O_1O_2 = \sqrt{R^2 + r^2 - r\sqrt{4R^2 + r^2}}.$$ Let A be an arbitrary point on k_1. The tangents from A to k_2 touch k_2 at B and C and intersect k_1 again at D and E, respectively. Prove that $BD \cdot CE = r^2$
2	Find all functions $f : R \to R$ such that $xf(x) - yf(y) = (x - y)f(x + y)$ for all $x, y \in R$.
3	Let p be a prime number, determine all positive integers (x, y, z) such that: $x^p + y^p = p^z$
–	Day 2
4	Let ABC be a triangle with incenter I, and let the tangency points of its incircle with its sides AB, BC, CA be C', A' and B' respectively. Prove that the circumcenters of AIA', BIB', and CIC' are collinear.
5	Let k be a positive integer and r_n be the remainder when $\binom{2n}{n}$ is divided by k. Find all k for which the sequence $(r_n)_{n=1}^{\infty}$ is eventually periodic.
6	Let n be a positive integer and A be a family of subsets of the set $\{1, 2, ..., n\}$, none of which contains another subset from A . Find the largest possible cardinality of A .

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