Saudi Arabia IMO Team Selection Test 2021

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– Day I

1 For a non-empty set T denote by $p(T)$ the product of all elements of T. Does there exist a set T of 2021 elements such that for any $a \in T$ one has that $P(T) - a$ is an odd integer? Consider two cases:
1) All elements of T are irrational numbers.
2) At least one element of T is a rational number.

2 Find all positive integers n, such that n is a perfect number and $\varphi(n)$ is power of 2.

[i]Note:a positive integer n, is called perfect if the sum of all its positive divisors is equal to $2n$.[/i]

3 Let ABC be a triangle with $AB < AC$, incenter I, and A excenter I_A. The incircle meets BC at D. Define $E = AD \cap BI_A$, $F = AD \cap CI_A$. Show that the circumcircle of $\triangle AID$ and $\triangle I_A EF$ are tangent to each other

– Day II

4 In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, \dots, Q_{24}$ whose corners are vertices of the 100-gon, so that

- the quadrilaterals $Q_1, \dots, Q_{24}$ are pairwise disjoint, and
- every quadrilateral Q_i has three corners of one color and one corner of the other color.

5 Let ABC be a non isosceles triangle with incenter I . The circumcircle of the triangle ABC has radius R. Let AL be the external angle bisector of $\angle BAC$with $L \in BC$. Let K be the point on perpendicular bisector of BC such that $IL \perp IK$.Prove that $OK = 3R$.

6 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f^{a^2+b^2}(a+b) = af(a) + bf(b)$$

for all integers a and b

– Day III

7 Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside side AB such that $AD < DB$. Let P and Q be two points inside sides BC and CA, respectively, such that $\angle DPB = \angle DQA = 90^\circ$. Let the perpendicular bisector of PQ meet line segment CQ at E, and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C. Suppose that P, E, F are collinear. Prove that $\angle ACB = 90^\circ$.

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8 The Fibonacci numbers $F_0, F_1, F_2, ...$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \ge 1$. Given an integer $n \ge 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, ..., n$ there exist some $x, y \in S$ such that $x - y = F_k$.

Proposed by Croatia

9

For a positive integer n, let $d(n)$ be the number of positive divisors of n, and let $\varphi(n)$ be the number of positive integers not exceeding n which are coprime to n. Prove that for any number C , there exist an integer n for which

$$\frac{\varphi(d(n))}{d(\varphi(n))} > C$$

variation was used in Saudi Arabia IMO TST

– Day IV

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10 Given a positive integer k show that there exists a prime p such that one can choose distinct integers $a_1, a_2 \cdots, a_{k+3} \in \{1, 2, \cdots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \cdots, k$.

South Africa

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11 Suppose that a, b, c, d are positive real numbers satisfying $(a + c)(b + d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Israel

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12 Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \dots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote $r(n)$ the fraction of integers $\{1, 2, \dots, n\}$ that are red.
Prove that there exists a positive integer $a \in \{1, 2, \dots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \dots, N$.

Netherlands