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– Day 1

Problem 1 Let $R > 0$, be an integer, and let $n(R)$ be the number um triples $(x,y,z) \in \mathbb{Z}^3$ such that $2x^2+3y^2+5z^2=R$. What is the value of

$\lim_{R\to\infty} \frac{n(1)+n(2)+\cdots+n(R)}{R^{3/2}}$?

Problem 2 For a positive integer a, define $F_1^{(a)}=1$, $F_2^{(a)}=a$ and for $n>2$, $F_n^{(a)}=F_{n-1}^{(a)}+F_{n-2}^{(a)}$. A positive integer is fibonatic when it is equal to $F_n^{(a)}$ for a positive integer a and $n>3$. Prove that there are infintely many not fibonatic integers.

Problem 3 Let $\mathbb{F}_{13} = \overline{0}, \overline{1}, \cdots, \overline{12}$ be the finite field with 13 elements (with sum and product modulus 13). Find how many matrix A of size 5 x 5 with entries in $\mathbb{F}_{13}$ exist such that

$$A^5 = I$$

where I is the identity matrix of order 5

– Day 2

Problem 4 For each of the following, provide proof or a counterexample:

a) Every 2×2 matrix with real entries can we written as the sum of the squares of two 2×2 matrices with real entries.
b) Every 3×3 matrix with real entries can we written as the sum of the squares of two 3×3 matrices with real entries.

Problem 5 Let N a positive integer.

In a spaceship there are $2\cdot N$ people, and each two of them are friends or foes (both relationships are symmetric). Two aliens play a game as follows:

1) The first alien chooses any person as she wishes.

2) Thenceforth, alternately, each alien chooses one person not chosen before such that the person chosen on each turn be a friend of the person chosen on the previous turn.

3) The alien that can't play in her turn loses.

Prove that second player has a winning strategy *if, and only if*, the $2\cdot N$ people can be divided in N pairs in such a way that two people in the same pair are friends.

Problem 6 Let $f(x) = 2x^2 + x - 1, f^0(x) = x$, and $f^{n+1}(x) = f(f^n(x))$ for all real $x > 0$ and $n \geq 0$ integer (that is, f^n is f iterated n times).

a) Find the number of distinct real roots of the equation $f^3(x) = x$
b) Find, for each $n \geq 0$ integer, the number of distinct real solutions of the equation $f^n(x) = 0$