IMC 2021

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–	Day 1 (August 3)
1	Let A be a real $n \times n$ matrix such that $A^3=0$ a) prove that there is unique real $n \times n$ matrix X that satisfied the equation $X+AX+XA^2=A$ b) Express X in terms of A
2	Let n and k be fixed positive integers , and a be arbitrary nonnegative integer . Choose a random k-element subset X of $\{1,2,...,k+a\}$ uniformly (i.e., all k-element subsets are chosen with the same probability) and, independently of X, choose random n-elements subset Y of $\{1,2,..,k+a+n\}$ uniformly. Prove that the probability $P\left(\text{min}(Y)>\text{max}(X)\right)$ does not depend on a.
3	We say that a positive real number d is $good$ if there exists an infinite squence $a_1,a_2,a_3,...\in (0,d)$ such that for each n, the points $a_1,a_2,...,a_n$ partition the interval $[0,d]$ into segments of length at most $\frac{1}{n}$ each . Find $\text{sup}\{d\|d \text{is good}\}$.
4	Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a function. Suppose that for every $\varepsilon >0$, there exists a function $g:\mathbb{R}\rightarrow (0,\infty)$ such that for every pair (x,y) of real numbers, if $\|x-y\|<\text{min}\{g(x),g(y)\}$, then $\|f(x)-f(y)\|<\varepsilon$ Prove that f is pointwise limit of a squence of continuous $\mathbb{R}\rightarrow\mathbb{R}$ functions i.e., there is a squence $h_1,h_2,...,$ of continuous $\mathbb{R}\rightarrow\mathbb{R}$ such that $\lim_{n\to\infty}h_n(x)=f(x)$ for every $x\in\mathbb{R}$
–	Day 2 (August 4)
5	Let A be a real $n\times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that $$2021B = A^m+B^2.$$ Prove that $\|\text{det} A\| \le 1$.
6	For a prime number p, let $GL_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\phi: GL_2(\mathbb{Z}/p\mathbb{Z})\rightarrow S_p$.

7 Let $D \subseteq \mathbb{C}$ be an open set containing the closed unit disk $\{z : |z| \leq 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$|f(0)| \leq \max_{|z|=1} |f(z)p(z)|$$

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8 Let n be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?
