**South African Mathematics Olympiad, Senior Round 3, 2021**

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by DylanN

**1** Find the smallest and largest integers with decimal representation of the form $ababa$ ($a \neq 0$) that are divisible by $11$.

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**2** Let $PAB$ and $PBC$ be two similar right-angled triangles (in the same plane) with $\angle PAB = \angle PBC = 90^\circ$ such that $A$ and $C$ lie on opposite sides of the line $PB$. If $PC = AC$, calculate the ratio $\frac{PA}{AB}$.

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**3** Determine the smallest integer $k > 1$ such that there exist $k$ distinct primes whose squares sum to a power of $2$.

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**4** Let $ABC$ be a triangle with $\angle ABC \neq 90^\circ$ and $AB$ its shortest side. Denote by $H$ the intersection of the altitudes of triangle $ABC$. Let $K$ be the circle through $A$ with centre $B$. Let $D$ be the other intersection of $K$ and $AC$. Let $K$ intersect the circumcircle of $BCD$ again at $E$. If $F$ is the intersection of $DE$ and $BH$, show that $BD$ is tangent to the circle through $D$, $F$, and $H$.

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**5** Determine all polynomials $a(x)$, $b(x)$, $c(x)$, $d(x)$ with real coefficients satisfying the simultaneous equations

$$b(x)c(x) + a(x)d(x) = 0$$
$$a(x)c(x) + (1 - x^2)b(x)d(x) = x + 1.$$

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**6** Jacob and Laban take turns playing a game. Each of them starts with the list of square numbers $1, 4, 9, \dots, 2021^2$, and there is a whiteboard in front of them with the number $0$ on it. Jacob chooses a number $x^2$ from his list, removes it from his list, and replaces the number $W$ on the whiteboard with $W + x^2$. Laban then does the same with a number from his list, and the repeat back and forth until both of them have no more numbers in their list. Now every time that the number on the whiteboard is divisible by $4$ after a player has taken his turn, Jacob gets a sheep. Jacob wants to have as many sheep as possible. What is the greatest number $K$ such that Jacob can guarantee to get at least $K$ sheep by the end of the game, no matter how Laban plays?

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