China Girls Math Olympiad 2021

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–	Day 1
1	Let $n \in \mathbb{N}^+$, $x_1, x_2, ..., x_{n+1}, p, q \in \mathbb{R}^+$, $p < q$ and $x_{n+1}^p > \sum_{i=1}^n x_i^p$. Prove that $(1) x_{n+1}^q > \sum_{i=1}^n x_i^q$; (2) $\left(x_{n+1}^p - \sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} < \left(x_{n+1}^q - \sum_{i=1}^n x_i^q\right)^{\frac{1}{q}}$.
2	In acute triangle ABC ($AB \neq AC$), I is its incenter and J is the A-excenter. X, Y are on minor arcs $\widehat{AB}$ and $\widehat{AC}$ respectively such that $\angle AXI = \angle AYJ = 90^\circ$. K is on line BC such that $KI = KJ$. Proof that line AK bisects $\overline{XY}$.
3	Find the smallest positive integer n, such that one can color every cell of a $n \times n$ grid in red, yellow or blue with all the following conditions satisfied: (1) the number of cells colored in each color is the same; (2) if a row contains a red cell, that row must contain a blue cell and cannot contain a yellow cell; (3) if a column contains a blue cell, it must contain a red cell but cannot contain a yellow cell.
4	Call a sequence of positive integers $(a_n)_{n\geq 1}$ a "CGMO sequence" if $(a_n)_{n\geq 1}$ strictly increases, and for all integers $n \geq 2022$, a_n is the smallest integer such that there exists a non-empty subset of $\{a_1, a_2, \cdots, a_{n-1}\}$ A_n where $a_n \cdot \prod\limits_{a\in A_n} a$ is a perfect square. Proof: there exists $c_1, c_2 \in \mathbb{R}^+$ s.t. for any "CGMO sequence" $(a_n)_{n\geq 1}$, there is a positive integer N that satisfies any $n \geq N$, $c_1 \cdot n^2 \leq a_n \leq c_2 \cdot n^2$.
–	Day 2
5	Proof that if 4 numbers (not necessarily distinct) are picked from $\{1, 2, \cdots, 20\}$, one can pick 3 numbers among them and can label these 3 as a, b, c such that $ax \equiv b \pmod{c}$ has integral solutions.
6	Given a finite set S, $P(S)$ denotes the set of all the subsets of S. For any $f : P(S) \to \mathbb{R}$,prove the following inequality: $$\sum_{A\in P(S)} \sum_{B\in P(S)} f(A)f(B)2^{\lvert A\cap B\rvert} \geq 0.$$

7 In an acute triangle ABC, $AB \neq AC$, O is its circumcenter. K is the reflection of B over AC and L is the reflection of C over AB. X is a point within ABC such that $AX \perp BC, XK = XL$. Points Y, Z are on $\overline{BK}, \overline{CL}$ respectively, satisfying $XY \perp CK, XZ \perp BL$.

Proof that B, C, Y, O, Z lie on a circle.

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8 Let m, n be positive integers, define: $f(x) = (x-1)(x^2-1)\cdots(x^m-1)$, $g(x) = (x^{n+1}-1)(x^{n+2}-1)\cdots(x^{n+m}-1)$.

Show that there exists a polynomial $h(x)$ of degree mn such that $f(x)h(x) = g(x)$, and its $mn+1$ coefficients are all positive integers.
