Centroamerican and Caribbean Math Olympiad 2021

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–	Day 1
1	An ordered triple (p, q, r) of prime numbers is called *parcera* if p divides $q^2 - 4$, q divides $r^2 - 4$ and r divides $p^2 - 4$. Find all parcera triples.
2	Let ABC be a triangle and let Γ be its circumcircle. Let D be a point on AB such that CD is parallel to the line tangent to Γ at A. Let E be the intersection of CD with Γ distinct from C, and F the intersection of BC with the circumcircle of $\triangle ADC$ distinct from C. Finally, let G be the intersection of the line AB and the internal bisector of $\angle DCF$. Show that $E,\ G,\ F$ and C lie on the same circle.
3	In a table consisting of 2021×2021 unit squares, some unit squares are colored black in such a way that if we place a mouse in the center of any square on the table it can walk in a straight line (up, down, left or right along a column or row) and leave the table without walking on any black square (other than the initial one if it is black). What is the maximum number of squares that can be colored black?
–	Day 2
4	There are 2021 people at a meeting. It is known that one person at the meeting doesn't have any friends there and another person has only one friend there. In addition, it is true that, given any 4 people, at least 2 of them are friends. Show that there are 2018 people at the meeting that are all friends with each other. *Note.* If A is friend of B then B is a friend of A.
5	Let $n \geq 3$ be an integer and $a_1, a_2, ..., a_n$ be positive real numbers such that m is the smallest and M is the largest of these numbers. It is known that for any distinct integers $1 \leq i, j, k \leq n$, if $a_i \leq a_j \leq a_k$ then $a_i a_k \leq a_j^2$. Show that $$a_1 a_2 \cdots a_n \geq m^2 M^{n-2}$$ and determine when equality holds
6	Let ABC be a triangle with $AB < AC$ and let M be the midpoint of AC. A point P (other than B) is chosen on the segment BC in such a way that $AB = AP$. Let D be the intersection of AC with the circumcircle of $\triangle ABP$ distinct from A, and E be the intersection of PM with the circumcircle of $\triangle ABP$ distinct from P. Let K be the intersection of lines AP and DE. Let F be

a point on BC (other than P) such that $KP = KF$. Show that $C,\ D,\ E$ and F lie on the same circle.
