Purple Comet Problems 2021

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– High School

1 The diagram shows two intersecting line segments that form some of the sides of two squares with side lengths 3 and 6. Two line segments join vertices of these squares. Find the area of the region enclosed by the squares and segments.
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png

2 A furniture store set the sticker price of a table 40 percent higher than the wholesale price that the store paid for the table. During a special sale, the table sold for 35 percent less than this sticker price. Find the percent the final sale price was of the original wholesale price of the table.

3 The diagram shows a semicircle with diameter 20 and the circle with greatest diameter that fits inside the semicircle. The area of the shaded region is $N\pi$, where N is a positive integer. Find N.
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4 A building contractor needs to pay his 108 workers $\$200$ each. He is carrying 122 one hundred dollar bills and 188 fifty dollar bills. Only 45 workers get paid with two $\$100$ bills. Find the number of workers who get paid with four $\$50$ bills.

5 There were three times as many red candies as blue candies on a table. After Darrel took the same number of red candies and blue candies, there were four times as many red candies as blue candies left on the table. Then after Cloe took 12 red candies and 12 blue candies, there were five times as many red candies as blue candies left on the table. Find the total number of candies that Darrel took.

6 A rectangular wooden block has a square top and bottom, its volume is 576, and the surface area of its vertical sides is 384. Find the sum of the lengths of all twelve of the edges of the block.

7 Among the 100 constants $a_1, a_2, a_3, ..., a_{100}$, there are 39 equal to -1 and 61 equal to $+1$. Find the sum of all the products $a_i a_j$, where $1 \leq i < j \leq 100$.

8 Pam lists the four smallest positive prime numbers in increasing order. When she divides the positive integer N by the first prime, the remainder is 1. When she divides N by the second prime, the remainder is 2. When she divides N by the third prime, the remainder is 3. When she divides N by the fourth prime, the remainder is 4. Find the least possible value for N.

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9 Find k such that $k\pi$ is the area of the region of points in the plane satisfying

$$\frac{x^2+y^2+1}{11} \le x \le \frac{x^2+y^2+1}{7}.$$

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10 A semicircle has diameter AB with $AB = 100$. Points C and D lie on the semicircle such that $AC = 28$ and $BD = 60$. Find CD.

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11 There are nonzero real numbers a and b so that the roots of $x^2 + ax + b$ are $3a$ and $3b$. There are relatively prime positive integers m and n so that $a - b = \frac{m}{n}$. Find $m + n$.

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12 Let L_1 and L_2 be perpendicular lines, and let F be a point at a distance 18 from line L_1 and a distance 25 from line L_2. There are two distinct points, P and Q, that are each equidistant from F, from line L_1, and from line L_2. Find the area of $\triangle FPQ$.

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13 Two infinite geometric series have the same sum. The first term of the first series is 1, and the first term of the second series is 4. The fifth terms of the two series are equal. The sum of each series can be written as $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

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14 Each of the cells of a 7×7 grid is painted with a color chosen randomly and independently from a set of N fixed colors. Call an edge hidden if it is shared by two adjacent cells in the grid that are painted the same color. Determine the least N such that the expected number of hidden edges is less than 3.

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15 Find the value of x where the graph of

$$y = \log_3(\sqrt{x^2+729}+x) - 2\log_3(\sqrt{x^2+729}-x)$$

crosses the x-axis.

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16 Paula rolls three standard fair dice. The probability that the three numbers rolled on the dice are the side lengths of a triangle with positive area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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17 For real numbers x let

$$f(x) = \frac{4^x}{25^{x+1}} + \frac{5^x}{2^{x+1}}.$$

Then $f\left(\frac{1}{1-\log_{10} 4}\right) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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18 The side lengths of a scalene triangle are roots of the polynomial

$$x^3 - 20x^2 + 131x - 281.3.$$

Find the square of the area of the triangle.

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19 Let a, b, c, d be an increasing arithmetic sequence of positive real numbers with common difference $\sqrt{2}$. Given that the product $abcd = 2021$, d can be written as $\frac{m+\sqrt{n}}{\sqrt{p}}$, where $m, n,$ and p are positive integers not divisible by the square of any prime. Find $m + n + p$.

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20 Let $ABCD$ be a convex quadrilateral with positive integer side lengths, $\angle A = \angle B = 120^\circ$, $|AD - BC| = 42$, and $CD = 98$. Find the maximum possible value of AB.

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21 Let a, b, and c be real numbers satisfying the equations

$$a^3 + abc = 26$$

$$b^3 + abc = 78$$

$$c^3 - abc = 104.$$

Find $a^3 + b^3 + c^3$.

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22 The least positive angle α for which

$$\left(\frac{3}{4} - \sin^2(\alpha)\right)\left(\frac{3}{4} - \sin^2(3\alpha)\right)\left(\frac{3}{4} - \sin^2(3^2\alpha)\right)\left(\frac{3}{4} - \sin^2(3^3\alpha)\right) = \frac{1}{256}$$

has a degree measure of $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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23 The sum

$$\sum_{k=3}^{\infty} \frac{1}{k(k^4 - 5k^2 + 4)^2}$$

is equal to $\frac{m^2}{2n^2}$, where m and n are relatively prime positive integers. Find $m + n$.

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24 Let x be a real number such that

$$4^{2x} + 2^{-x} + 1 = (129 + 8\sqrt{2})(4^x + 2^{-x} - 2^x).$$

Find $10x$.

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25 The area of the triangle whose altitudes have lengths 36.4, 39, and 42 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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26 The product

$$\left(\frac{1}{2^3-1}+\frac{1}{2}\right)\left(\frac{1}{3^3-1}+\frac{1}{2}\right)\left(\frac{1}{4^3-1}+\frac{1}{2}\right)\cdots\left(\frac{1}{100^3-1}+\frac{1}{2}\right)$$

can be written as $\frac{r}{s2^t}$ where r, s, and t are positive integers and r and s are odd and relatively prime. Find $r+s+t$.

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27 Let $ABCD$ be a cyclic quadrilateral with $AB=5$, $BC=10$, $CD=11$, and $DA=14$. The value of $AC+BD$ can be written as $\frac{n}{\sqrt{pq}}$, where n is a positive integer and p and q are distinct primes. Find $n+p+q$.

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28 Let z_1, z_2, z_3, $\cdots$, z_{2021} be the roots of the polynomial $z^{2021}+z-1$. Evaluate

$$\frac{z_1^3}{z_1+1}+\frac{z_2^3}{z_2+1}+\frac{z_3^3}{z_3+1}+\cdots+\frac{z_{2021}^3}{z_{2021}+1}.$$

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29 Two cubes with edge length 3 and two cubes with edge length 4 sit on plane P so that the four cubes share a vertex, and the two larger cubes share no faces with each other as shown below. The cube vertices that do not touch P or any of the other cubes are labeled A, B, C, D, E, F, G, and H. The four cubes lie inside a right rectangular pyramid whose base is on P and whose triangular sides touch the labeled vertices with one side containing vertices A, B, and C, another side containing vertices D, E, and F, and the two other sides each contain one of G and H. Find the volume of the pyramid.
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30 For positive integer k, define $x_k = 3k+\sqrt{k^2-1}-2(\sqrt{k^2-k}+\sqrt{k^2+k})$. Then $\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{1681}}=\sqrt{m}-n$, where m and n are relatively prime positive integers. Find $m+n$.

– Middle School

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1 The diagram below shows two concentric circles whose areas are 7 and 53 and a pair of perpendicular lines where one line contains diameters of both circles and the other is tangent to the smaller circle. Find the area of the shaded region.
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2 At one school, 85 percent of the students are taking mathematics courses, 55 percent of the students are taking history courses, and 7 percent of the students are taking neither mathematics nor history courses. Find the percent of the students who are taking both mathematics and history courses.

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3 Let M and m be, respectively, the greatest and the least ten-digit numbers that are rearrangements of the digits 0 through 9 such that no two adjacent digits are consecutive. Find $M - m$.

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4 The diagram shows a regular pentagon $ABCDE$ and a square $ABFG$. Find the degree measure of $\angle FAD$.
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5 Ted is fi ve times as old as Rosie was when Ted was Rosie's age. When Rosie reaches Ted's current age, the sum of their ages will be 72. Find Ted's current age.

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6 Find the least integer $n > 60$ so that when $3n$ is divided by 4, the remainder is 2 and when $4n$ is divided by 5, the remainder is 1.

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7 Find the sum of all positive integers x such that there is a positive integer y satisfying $9x^2 - 4y^2 = 2021$.

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8 Fiona had a solid rectangular block of cheese that measured 6 centimeters from left to right, 5 centimeters from front to back, and 4 centimeters from top to bottom. Fiona took a sharp knife and sliced off a 1 centimeter thick slice from the left side of the block and a 1 centimeter slice from the right side of the block. After that, she sliced off a 1 centimeter thick slice from the front side of the remaining block and a 1 centimeter slice from the back side of the remaining block. Finally, Fiona sliced off a 1 centimeter slice from the top of the remaining block and a 1 centimeter slice from the bottom of the remaining block. Fiona now has 7 blocks of cheese. Find the total surface area of those seven blocks of cheese measured in square centimeters.

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9 Let a and b be positive real numbers satisfying

$$a - 12b = 11 - \frac{100}{a} \quad and \quad a - \frac{12}{b} = 4 - \frac{100}{a}.$$

Then $a + b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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10 Find the value of n such that the two inequalities

$$|x + 47| \le n \quad and \quad \frac{1}{17} \le \frac{4}{3 - x} \le \frac{1}{8}$$

have the same solutions.

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11 Find the minimum possible value of —m -n—, where m and n are integers satisfying $m + n = mn - 2021$.

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12 A farmer wants to create a rectangular plot along the side of a barn where the barn forms one side of the rectangle and a fence forms the other three sides. The farmer will build the fence by tting together 75 straight sections of fence which are each 4 feet long. The farmer will build the fence to maximize the area of the rectangular plot. Find the length in feet along the side of the barn of this rectangular plot.

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13 Find the greatest prime number p such that p^3 divides

$$\frac{122!}{121} + 123! :$$

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14 In base ten, the number $100! = 100 \cdot 99 \cdot 98 \cdot 97 ... 2 \cdot 1$ has 158 digits, and the last 24 digits are all zeros. Find the number of zeros there are at the end of this same number when it is written in base 24.

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15 Let m and n be positive integers such that

$$(m^3 - 27)(n^3 - 27) = 27(m^2n^2 + 27) :$$

Find the maximum possible value of $m^3 + n^3$.

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16 Find the number of distinguishable groupings into which you can place 3 indistinguishable red balls and 3 indistinguishable blue balls. Here the groupings $RR - BR - B - B$ and $B - RB - B - RR$ are indistinguishable because the groupings are merely rearranged, but $RRB - BR - B$ is distinguishable from $RBB - BR - R$.

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17 Points X and Y lie on side $\overline{AB}$ of $\triangle\ ABC$ such that $AX = 20$, $AY = 28$, and $AB = 42$. Suppose $XC = 26$ and $YC = 30$. Find $AC + BC$.

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18 Three red books, three white books, and three blue books are randomly stacked to form three piles of three books each. The probability that no book is the same color as the book immediately on top of it is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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19 For some integers u,v, and w, the equation

$$26ab - 51bc + 74ca = 12(a^2 + b^2 + c^2)$$

holds for all real numbers a, b, and c that satisfy

$$au + bv + cw = 0$$

Find the minimum possible value of $u^2 + v^2 + w^2$.

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20 Square $ABCD$ with side length 2 begins in position 1 with side AD horizontal and vertex A in the lower right corner. The square is rotated 90^o clockwise about vertex A into position 2 so that vertex D ends up where vertex B was in position 1. Then the square is rotated 90^o clockwise about vertex C into position 3 so that vertex B ends up where vertex D was in position 2 and vertex B was in position 1, as shown below. The area of the region of points in the plane that were covered by the square at some time during its rotations can be written $\frac{p\pi+\sqrt{q}+r}{s}$, where $p, q, r,$ and s are positive integers, and p and s are relatively prime. Find $p+q+r+s$.
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