XI Iberoamerican Interuniversitary Mathematics Competition - Colombia

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by Ozc

Problem 1 Determine all triples of integers (x, y, z) that satisfy the equation

$$x^z + y^z = z.$$

Problem 2 Consider the set

$$\{0,1\}^n = \{X = (x_1, x_2, \dots, x_n) : x_i \in \{0,1\}, 1 \leq i \leq n\}.$$

We say that $X > Y$ if $X \neq Y$ and the following n inequalities are satisfy

$$x_1 \geq y_1, x_1 + x_2 \geq y_1 + y_2, \dots, x_1 + x_2 + \cdots + x_n \geq y_1 + y_2 + \cdots + y_n.$$

We define a chain of length k as a subset $Z_1, \dots, Z_k \subseteq \{0,1\}^n$ of distinct elements such that $Z_1 > Z_2 > \cdots > Z_k$.

Determine the lenght of longest chain in $\{0,1\}^n$.

Problem 3 Let $\{a_n\}_{n\in\mathbb{N}}$ a sequence of non zero real numbers.

For $m \geq 1$, we define:

$$X_m = \left\{ X \subseteq \{0, 1, \dots, m-1\} : \left| \sum_{x \in X} a_x \right| > \frac{1}{m} \right\}.$$

Show that

$$\lim_{n\to\infty} \frac{|X_n|}{2^n} = 1.$$

Problem 4 Let $(G, *)$ a group of $n > 1$ elements, and let $g \in G$ be an element distinct from the identity.

Ana and Bob play with the group G on the following way:

Starting with Ana and playing alternately, each player selects an element of G that has not been selected before, until each element of G have been selected or a player have selected the elements a and $a * g$ for some $a \in G$.

In that case it is said that the player loses and his opponent wins.

$a)$ If n is odd, show that, independent of element g, one of the two players has a winning strategy and determines which player possesses such a strategy.

$b)$ If n is even, show that there exists an element $g \in G$ for which none of the players has a winning strategy.

Note: A group $(G, *)$ es a set G together with a binary operation $* : G \times G \to G$ that satisfy the following properties (i) $*$ is asociative: $\forall a, b, c \in G (a * b) * c = a * (b * c)$; (ii) there exists an identity element $e \in G$ such that $\forall a \in G, a * e = e * a = a$; (iii) there exists inverse elements: $\forall a \in G \exists a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

Problem 5 Let $\{k_1, k_2, \dots, k_m\}$ a set of m integers. Show that there exists a matrix $m \times m$ with integers entries A such that each of the matrices $A + k_j I, 1 \leq j \leq m$ are invertible and their entries have integer entries (here I denotes the identity matrix).

Problem 6 Determine all the injective functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$, such that for each pair of integers (m, n) the following conditions hold:

$a)$ $f(mn) = f(m)f(n)$ $b)$ $f(m^2 + n^2) \mid f(m^2) + f(n^2)$.
