Saudi Arabia Team Selection Test for Balkan Math Olympiad 2021

www.artofproblemsolving.com/community/c2457704

by parmenides51

– Day I

1 Do there exist two polynomials P and Q with integer coefficient such that
i) both P and Q have a coefficient with absolute value bigger than 2021,
ii) all coefficients of $P \cdot Q$ by absolute value are at most 1.

2 Let ABC be an acute, non-isosceles triangle with H the orthocenter and M the midpoint of AH. Denote O_1,O_2 as the centers of circles pass through H and respectively tangent to BC at B, C. Let X, Y be the ex-centers which respect to angle H in triangles HMO_1,HMO_2. Prove that XY is parallel to O_1O_2.

3 Let x, y and z be odd positive integers such that $\gcd(x, y, z) = 1$ and the sum $x^2 + y^2 + z^2$ is divisible by $x + y + z$. Prove that $x + y + z - 2$ is not divisible by 3.

4 In the popular game of Minesweeper, some fields of an $a \times b$ board are marked with a mine and on all the remaining fields the number of adjacent fields that contain a mine is recorded. Two fields are considered adjacent if they share a common vertex. For which $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ is it possible for some a and b , $ab > 2021$, to create a board whose fields are covered in mines, except for 2021 fields who are all marked with k?

– Day II

1 There are $n \geq 2$ positive integers written on the whiteboard. A move consists of three steps: calculate the least common multiple N of all numbers then choose any number a and replace a by N/a .
Prove that, using a finite number of moves, you can always make all the numbers on the whiteboard equal to 1.

2 Let ABC be an acute triangle with $AB < AC$ and inscribed in the circle (O). Denote I as the incenter of ABC and D, E as the intersections of AI with BC, (O) respectively. Take a point K on BC such that $\angle AIK = 90^o$ and KA, KE meet (O) again at M,N respectively. The rays ND, NI meet the circle (O) at Q,P.
1. Prove that the quadrilateral $MPQE$ is a kite.
2. Take J on IO such that $AK \perp AJ$. The line through I and perpendicular to OI cuts BC at R ,cuts EK at S .Prove that $OR \parallel JS$.

3 Let a, b, and c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$$

---

4 A set of n points in space is given, no three of which are collinear and no four of which are co-planar (on a single plane), and each pair of points is connected by a line segment. Initially, all the line segments are colorless. A positive integer b is given and Alice and Bob play the following game. In each turn Alice colors one segment red and then Bob colors up to b segments blue. This is repeated until there are no more colorless segments left. If Alice colors a red triangle, Alice wins. If there are no more colorless segments and Alice hasn't succeeded in coloring a red triangle, Bob wins. Neither player is allowed to color over an already colored line segment.
1. Prove that if $b < \sqrt{2n-2} - \frac{3}{2}$, then Alice has a winning strategy.
2. Prove that if $b \geq 2\sqrt{n}$, then Bob has a winning strategy.
