Middle European Mathematical Olympiad 2021

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– Individual Competition

1 Determine all real numbers A such that every sequence of non-zero real numbers $x_1, x_2, \ldots$ satisfying

$$x_{n+1} = A - \frac{1}{x_n}$$

for every integer $n \geq 1$, has only finitely many negative terms.

2 Let m and n be positive integers. Some squares of an $m \times n$ board are coloured red. A sequence $a_1, a_2, \ldots, a_{2r}$ of $2r \geq 4$ pairwise distinct red squares is called a *bishop circuit* if for every $k \in \{1, \ldots, 2r\}$, the squares a_k and a_{k+1} lie on a diagonal, but the squares a_k and a_{k+2} do not lie on a diagonal (here $a_{2r+1} = a_1$ and $a_{2r+2} = a_2$).
In terms of m and n, determine the maximum possible number of red squares on an $m \times n$ board without a bishop circuit.
(*Remark*. Two squares lie on a diagonal if the line passing through their centres intersects the sides of the board at an angle of $45°$.)

3 Let ABC be an acute triangle and D an interior point of segment BC. Points E and F lie in the half-plane determined by the line BC containing A such that DE is perpendicular to BE and DE is tangent to the circumcircle of ACD, while DF is perpendicular to CF and DF is tangent to the circumcircle of ABD. Prove that the points A, D, E and F are concyclic.

4 Let $n \geq 3$ be an integer. Zagi the squirrel sits at a vertex of a regular n-gon. Zagi plans to make a journey of $n-1$ jumps such that in the i-th jump, it jumps by i edges clockwise, for $i \in \{1, \ldots, n-1\}$. Prove that if after $\lceil \frac{n}{2} \rceil$ jumps Zagi has visited $\lceil \frac{n}{2} \rceil + 1$ distinct vertices, then after $n-1$ jumps Zagi will have visited all of the vertices.
(*Remark*. For a real number x, we denote by $\lceil x \rceil$ the smallest integer larger or equal to x.)

– Team Competition

1 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the inequality

$$f(x^2) - f(y^2) \leq (f(x) + y)(x - f(y))$$

holds for all real numbers x and y.

2 Given a positive integer n, we say that a polynomial P with real coefficients is n-pretty if the equation $P(\lfloor x \rfloor) = \lfloor P(x) \rfloor$ has exactly n real solutions. Show that for each positive integer n

- there exists an n-pretty polynomial;
- any n-pretty polynomial has a degree of at least $\frac{2n+1}{3}$.

(*Remark*. For a real number x, we denote by $\lfloor x \rfloor$ the largest integer smaller than or equal to x.)

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3 Let n, b and c be positive integers. A group of n pirates wants to fairly split their treasure. The treasure consists of $c \cdot n$ identical coins distributed over $b \cdot n$ bags, of which at least $n-1$ bags are initially empty. Captain Jack inspects the contents of each bag and then performs a sequence of moves. In one move, he can take any number of coins from a single bag and put them into one empty bag. Prove that no matter how the coins are initially distributed, Jack can perform at most $n-1$ moves and then split the bags among the pirates such that each pirate gets b bags and c coins.

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4 Let n be a positive integer. Prove that in a regular $6n$-gon, we can draw $3n$ diagonals with pairwise distinct ends and partition the drawn diagonals into n triplets so that:

- the diagonals in each triplet intersect in one interior point of the polygon and
- all these n intersection points are distinct.

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5 Let AD be the diameter of the circumcircle of an acute triangle ABC. The lines through D parallel to AB and AC meet lines AC and AB in points E and F, respectively. Lines EF and BC meet at G. Prove that AD and DG are perpendicular.

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6 Let ABC be a triangle and let M be the midpoint of the segment BC. Let X be a point on the ray AB such that $2\angle CXA = \angle CMA$. Let Y be a point on the ray AC such that $2\angle AYB = \angle AMB$. The line BC intersects the circumcircle of the triangle AXY at P and Q, such that the points P, B, C, and Q lie in this order on the line BC. Prove that $PB = QC$.

Proposed by Dominik Burek, Poland

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7 Find all pairs (n, p) of positive integers such that p is prime and

$$1 + 2 + \cdots + n = 3 \cdot (1^2 + 2^2 + \cdot + p^2).$$

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8 Prove that there are infinitely many positive integers n such that n^2 written in base 4 contains only digits 1 and 2.
