Balkan MO 2021

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1	Let ABC be a triangle with $AB < AC$. Let ω be a circle passing through B, C and assume that A is inside ω. Suppose X, Y lie on ω such that $\angle BXA = \angle AYC$. Suppose also that X and C lie on opposite sides of the line AB and that Y and B lie on opposite sides of the line AC. Show that, as X, Y vary on ω, the line XY passes through a fixed point. *Proposed by Aaron Thomas, UK*
2	Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that $f(x + f(x) + f(y)) = 2f(x) + y$ for all positive reals x, y. *Proposed by Athanasios Kontogeorgis, Greece*
3	Let a, b and c be positive integers satisfying the equation $(a, b) + [a, b] = 2021^c$. If $\lvert a - b \rvert$ is a prime number, prove that the number $(a + b)^2 + 4$ is composite. *Proposed by Serbia*
4	Problem 4. Angel has a warehouse, which initially contains 100 piles of 100 pieces of rubbish each. Each morning, Angel performs exactly one of the following moves: (a) He clears every piece of rubbish from a single pile. (b) He clears one piece of rubbish from each pile. However, every evening, a demon sneaks into the warehouse and performs exactly one of the following moves: (a) He adds one piece of rubbish to each non-empty pile. (b) He creates a new pile with one piece of rubbish. What is the first morning when Angel can guarantee to have cleared all the rubbish from the warehouse?