Czech-Polish-Slovak Junior Match 2021

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by parmenides51

–	Individual
1	You are given a 2×2 array with a positive integer in each field. If we add the product of the numbers in the first column, the product of the numbers in the second column, the product of the numbers in the first row and the product of the numbers in the second row, we get 2021. a) Find possible values for the sum of the four numbers in the table. b) Find the number of distinct arrays that satisfy the given conditions that contain four pairwise distinct numbers in arrays.
2	An acute triangle ABC is given. Let us denote by D and E the orthogonal projections, respectively of points B and C on the bisector of the external angle BAC. Let F be the point of intersection of the lines BE and CD. Show that the lines AF and DE are perpendicular.

3	A *cross* is the figure composed of 6 unit squares shown below (and any figure made of it by rotation).
	`https://cdn.artofproblemsolving.com/attachments/6/0/6d4e0579d2e4c4fa67fd1219837576189ec9c`
	`png`
	Find the greatest number of crosses that can be cut from a 6×11 divided sheet of paper into unit squares (in such a way that each cross consists of six such squares).

4	Find the smallest value that the expression takes $x^4 + y^4 - x^2y - xy^2$, for positive numbers x and y satisfying $x + y \le 1$.
5	A regular heptagon $ABCDEFG$ is given. The lines AB and CE intersect at P. Find the measure of the angle $\angle PDG$.
–	Team
1	Consider a trapezoid $ABCD$ with bases AB and CD satisfying $\|AB\| > \|CD\|$. Let M be the midpoint of AB. Let the point P lie inside $ABCD$ such that $\|AD\| = \|PC\|$ and $\|BC\| = \|PD\|$. Prove that if $\|\angle CMD\| = 90^o$, then the quadrilaterals $AMPD$ and $BMPC$ have the same area.
2	Let the numbers $x_i \in \{-1, 1\}$ be given for $i = 1, 2, ..., n$, satisfying $$x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1 = 0.$$ Prove that n is divisible by 4.

3 Find the number of pairs (a, b) of positive integers with the property that the greatest common divisor of a and b is equal to $1 \cdot 2 \cdot 3 \cdot ... \cdot 50$, and the least common multiple of a and b is $1^2 \cdot 2^2 \cdot 3^2 \cdot ... \cdot 50^2$.

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4 Find the smallest positive integer n with the property that in the set $\{70, 71, 72, ...70 + n\}$ you can choose two different numbers whose product is the square of an integer.

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5 Find all three real numbers (x, y, z) satisfying the system of equations

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{x}{z} + \frac{z}{y} + \frac{y}{x}$$

$$x^2 + y^2 + z^2 = xy + yz + zx + 4$$

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6 Let $s(n)$ denote the sum of digits of a positive integer n. Using six different digits, we formed three 2-digits p, q, r such that

$$p \cdot q \cdot s(r) = p \cdot s(q) \cdot r = s(p) \cdot q \cdot r.$$

Find all such numbers p, q, r.
