Moldova Team Selection Test 2021

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– Day 1

1 Let $P(x)=x^3+ax^2+bx+1$ be a polynomial with real coefficients and three real roots ρ_1, ρ_2, ρ_3 such that $|\rho_1|<|\rho_2|<|\rho_3|$. Let A be the point where the graph of $P(x)$ intersects yy' and the point $B(\rho_1,0)$, $C(\rho_2,0)$, $D(\rho_3,0)$. If the circumcircle of $\triangle ABD$ intersects yy' for a second time at E, find the minimum value of the length of the segment EC and the polynomials for which this is attained.

Brazitikos Silouanos, Greece

2 Prove that if p and q are two prime numbers, such that

$$p+p^2+p^3+...+p^q=q+q^2+q^3+...+q^p,$$

then $p=q$.

3 Acute triangle ABC with $AB>BC$ is inscribed in circle Ω. Points D and E, that lie on (BC) and (AB) are the feet of altitudes from A and C in triangle ABC, and M is the midpoint of the segment DE. Half-line $(AM$ intersects the circle Ω for the second time in N. Show that the circumcenter of triangle MDN lies on the line BC.

4 Let n be a positive integer. A panel of dimenisions $2n\times 2n$ is divided in $4n^2$ squares with dimensions 1×1. What is the highest possible number of diagonals that can be drawn in 1×1 squares, such that each two diagonals have no common points.

– Day 2

5 Let ABC be an equilateral triangle. Find all positive integers n, for which the function f, defined on all points M from the circle S circumscribed to triangle ABC, defined by the formula $f:S\rightarrow R, f(M)=MA^n+MB^n+MC^n$, is a constant function.

6 There are 14 players participating at a chess tournament, each playing one game with every other player. After the end of the tournament, the players were ranked in descending order based on their points. The sum of the points of the first three players is equal with the sum of the points of the last nine players. What is the highest possible number of draws in the tournament.(For a victory the player gets 1 point, for a loss 0 points, in a draw both players get $0,5$ points.)

7 Positive real numbers a, b, c satisfy $a+b+c=1$. Show that

$$\frac{a+1}{\sqrt{a+bc}}+\frac{b+1}{\sqrt{b+ca}}+\frac{c+1}{\sqrt{c+ab}}\geq\frac{2}{a^2+b^2+c^2}.$$

When does the equality take place?

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8 Determine all positive integers n such that $\frac{a^2+n^2}{b^2-n^2}$ is a positive integer for some $a,b \in \mathbb{N}$. $Turkey$

– Day 3

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9 Positive real numbers a, b, c satisfy $a+b+c=1$. Find the smallest possible value of

$$E(a,b,c)=\frac{a^3}{1-a^2}+\frac{b^3}{1-b^2}+\frac{c^3}{1-c^2}.$$

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10 On a board there are written the integers from 1 to 119. Two players, A and B, make a move by turn. A $move$ consists in erasing 9 numbers from the board. The player after whose move two numbers remain on the board wins and his score is equal with the positive difference of the two remaining numbers. The player A makes the first move. Find the highest integer k, such that the player A can be sure that his score is not smaller than k.

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11 In a convex quadrilateral $ABCD$ the angles BAD and BCD are equal. Points M and N lie on the sides (AB) and (BC) such that the lines MN and AD are parallel and $MN=2AD$. The point H is the orthocenter of the triangle ABC and the point K is the midpoint of MN. Prove that the lines KH and CD are perpendicular.

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12 Prove that $n!\cdot(n+1)!\cdot(n+2)!$ divides $(3n)!$ for every integer $n\geq 3$.
