Final Round - Costa Rica 2017

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– Day 1

1 Let the regular hexagon $ABCDEF$ be inscribed in a circle with center O, N be such a point Let $E-N-C$, M a point such that $A-M-C$ and R a point on the circumference, such that $D-N-R$. If $\angle EFR = 90^o$, $\frac{AM}{AC}=\frac{CN}{EC}$ and $AC=\sqrt3$, calculate AM.

Notation: $A-B-C$ means than points A, B, C are collinear in that order i.e. B lies between A and C.

2 Determine the greatest common divisor of the numbers:

$$5^5-5, 7^7-7, 9^9-9, ..., 2017^{2017}-2017,$$

3 A game consists of a grid of 4×4 and tiles of two colors (Yellow and White). A player chooses a type of token and gives it to the second player who places it where he wants, then the second player chooses a type of token and gives it to the first who places it where he wants, They continue in this way and the one who manages to form a line with three tiles of the same color wins (horizontal, vertical or diagonal and regardless of whether it is the tile you started with or not). Before starting the game, two yellow and two white pieces are already placed as shows the figure below.
`https://cdn.artofproblemsolving.com/attachments/b/5/ba11377252c278c4154a8c3257faf363430ef
png`
Yolanda and Xinia play a game. If Yolanda starts (choosing the token and giving it to Xinia for this to place) indicate if there is a winning strategy for either of the two players and, if any, describe the strategy.

– Day 2

4 Let k be a real number, such that the equation $kx^2+k=3x^2+2-2kx$ has two real solutions different. Determine all possible values of k, such that the sum of the roots of the equation is equal to the product of the roots of the equation increased by k.

5 Consider two circles Π_1 and Π_1 tangent externally at point S, such that the radius of Π_2 is triple the radius of Π_1. Let ℓ be a line that is tangent to Π_1 at point P and tangent to Π_2 at point Q, with P and Q different from S. Let T be a point at Π_2, such that the segment TQ is diameter of Π_2 and let point R be the intersection of the bisector of $\angle SQT$ with ST. Prove that $QR=RT$.

6 Let $f :]0.\infty[\to R$ be a strictly increasing function, such that

$$f(x)f\left(f(x)+\frac{1}{x}\right)=1.$$

Find $f(1)$.

– Shortlist

A1 Let $P(x)$ be a polynomial of degree $2n$, such that $P(k)=\frac{k}{k+1}$ for $k=0,...,2n$. Determine $P(2n+1)$.

F1 Let $f:Z^+\to R$, such that $f(1)=2018$ and $f(1)+f(2)+...+f(n)=n^2f(n)$, for all $n>1$. Find the value $f(2017)$.

G2 Consider the right triangle $\triangle\ ABC$ right at A and let D be a point on the hypotenuse BC. Consider the line that passes through the incenters of $\triangle\ ABD$ and $\triangle\ ACD$, and let K and L the intersections of said line with AB and AC respectively. Show that if $AK=AL$ then D is the foot of the altitude on the hypotenuse.

G4 In triangle ABC with incenter I and circumcircle ω, the tangent through C to ω intersects AB at point D. The angle bisector of $\angle CDB$ intersects AI and BI at E and F, respectively. Let M be the midpoint of $[EF]$. Prove that line MI passes through the midpoint of arc ACB of w .

LR2 There is a set of 17 consecutive positive integers. Let m be the smallest of these numbers. Determine for which values of m the set can be divided into three subsets disjoint, such that the sum of the elements of each subset is the same.

N2 A positive integer is said to be ”nefelibata” if, upon taking its last digit and placing it as the first digit, keeping the order of all the remaining digits intact (for example, 312 -¿ 231), the resulting number is exactly double the original number. Find the smallest possible nefelibata number.