Final Round - Costa Rica 2018

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– Day 1

1 There are 10 points on a circle and all possible segments are drawn on the which two of these points are the endpoints. Determine the probability that selecting two segments randomly, they intersect at some point (it could be on the circumference).

2 Let a, b, c, and d be real numbers. The six sums of two numbers x and y, different from the previous four, are 117, 510, 411, 252, in no particular order. Determine the maximum possible value of $x + y$.

3 In the attached figure, point C is the center of the circle, AB is tangent to the circle, $P - C - P'$ and $AC \perp PP'$. If $AT = 2$ cm. and $AB = 4$ cm, calculate BQ
https://cdn.artofproblemsolving.com/attachments/e/e/d47429b82fb87299c40f5224489313909cfd0
png
Notation: $A - B - C$ means than points A, B, C are collinear in that order i.e. B lies between A and C.

– Day 2

4 Determine if there exists a function f: $N^* \to N^*$ that satisfies that for all $n \in N^*$,

$$10^{f(n)} < 10n + 1 < 10^{f(n)+1}.$$

Justify your answer.

Note: N^* denotes the set of positive integers.

5 Let a and b be even numbers, such that $M = (a+b)^2 - ab$ is a multiple of 5. Consider the following statements:
I) The unit digits of a^3 and b^3 are different.
II) M is divisible by 100.
Please indicate which of the above statements are true with certainty.

6 The four faces of a right triangular pyramid are equilateral triangles whose edge measures 3 dm. Suppose the pyramid is hollow, resting on one of its faces at a horizontal surface (see attached figure) and that there is 2 dm^3 of water inside. Determine the height that the liquid reaches inside the pyramid.
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png

– Shortlist

A1 If $x \in R - \{-7\}$, determine the smallest value of the expression

$$\frac{2x^2+98}{(x+7)^2}$$

A2 Determine the sum of the real roots of the equation

$$x^2-8x+20=2\sqrt{x^2-8x+30}$$

F2 Consider $f(n,m)$ the number of finite sequences of 1's and 0's such that each sequence that starts at 0, has exactly n 0's and m 1's, and there are not three consecutive 0's or three 1's. Show that if $m, n > 1$, then

$$f(n,m)=f(n-1,m-1)+f(n-1,m-2)+f(n-2,m-1)+f(n-2,m-2)$$

F3 Consider a function $f : R \to R$ that fulfills the following two properties: f is periodic of period 5 (that is, for all $x \in R$, $f(x+5) = f(x)$), and by restricting f to the interval $[-2, 3]$, f coincides to x^2. Determine the value of $f(2018)$.

G1 Let O be the center of the circle circumscribed to $\triangle\ ABC$, and let P be any point on BC ($P \ne B$ and $P \ne C$). Suppose that the circle circumscribed to $\triangle\ BPO$ intersects AB at R ($R \ne A$ and $R \ne B$) and that the circle circumscribed to $\triangle\ COP$ intersects CA at point Q ($Q \ne C$ and $Q \ne A$).
1) Show that $\triangle\ PQR \sim\triangle\ ABC$ and that$O$ is orthocenter of $\triangle\ PQR$.
2) Show that the circles circumscribed to the triangles $\triangle\ BPO$, $\triangle\ COP$, and $\triangle\ PQR$ all have the same radius.

G2 Consider $\triangle\ ABC$, with AD bisecting $\angle BAC$, D on segment BC. Let E be a point on BC, such that $BD = EC$. Through E we draw the line ℓ parallel to AD and consider a point P on it and inside the $\triangle\ ABC$. Let G be the point where line BP cuts side AC and let F be the point where line CP to side AB. Show that $BF = CG$.

G5 In the accompanying figure, semicircles with centers$A$ and B have radii 4 and 2, respectively. Furthermore, they are internally tangent to the circle of diameter PQ. Also the semicircles with centers A and B are externally tangent to each other. The circle with center C is internally tangent to the semicircle with diameter PQ and externally tangent to the others two semicircles. Determine the value of the radius of the circle with center C.

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LRP1 Arnulfo and Berenice play the following game: One of the two starts by writing a number from 1 to 30, the other chooses a number from 1 to 30 and adds it to the initial number, the first player chooses a number from 1 to 30 and adds it to the previous result, they continue doing the same until someone manages to add 2018. When Arnulfo was about to start, Berenice told him that it was unfair, because whoever started had a winning strategy, so the numbers had better change.
So they asked the following question:
Adding chosen numbers from 1 to a, until reaching the number b, what conditions must meet a and b so that the first player does not have a winning strategy?
Indicate if Arnulfo and Berenice are right and answer the question asked by them.

LRP3 Jordan is in the center of a circle whose radius is 100 meters and can move one meter at a time, however, there is a giant who at every step can force you to move in the opposite direction to the one he chose (it does not mean returning to the place of departure, but advance but in the opposite direction to the chosen one). Determine the minimum number of steps that Jordan must give to get out of the circle.

LRP4 On a 30×30 board both rows 1 to 30 and columns are numbered, in addition, to each box is assigned the number ij, where the box is in row i and column j. N columns and m rows are chosen, where $1 < n$ and $m < 30$, and the cells that are simultaneously in any of the rows and in any of the selected columns are painted blue. They paint the others red .
(a) Prove that the sum of the numbers in the blue boxes cannot be prime.
(b) Can the sum of the numbers in the red cells be prime?

LRP5 The Matini company released a special album with the flags of the 12 countries that compete in the CONCACAM Mathematics Cup. Each postcard envelope has two flags chosen randomly. Determine the minimum number of envelopes that need to be opened to that the probability of having a repeated flag is 50%.

N1 Prove that there are only two sets of consecutive positive integers that satisfy that the sum of its elements is equal to 100.

N2 Determine all triples (a, b, c) of nonnegative integers that satisfy:

$$(c-1)(ab-b-a) = a+b-2$$

N3 Let a and b be positive integers such that $2a^2 + a = 3b^2 + b$. Prove that $a - b$ is a perfect square.

N4 Let p be a prime number such that $p = 10^{d-1} + 10^{d-2} + ... + 10 + 1$. Show that d is a prime.