Final Round - Costa Rica 2016
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–	Day 1 - Shortlist
A1	Prove that $$\left(\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+...+\frac{1}{\sqrt{2015}+\sqrt{2016}}\right)^2(2017+24\sqrt{14})=2015^2$$
A2	Find all integer solutions of the equation $p(x+y)=xy$, where p is a prime number.
A3	Let x and y be two positive real numbers, such that $x+y=1$. Prove that $$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\ge 9$$
G1	Let $\triangle\ ABC$ be isosceles with $AB=AC$. Let ω be its circumscribed circle and O its circumcenter. Let D be the second intersection of CO with ω. Take a point E in AB such that $DE \parallel AC$ and suppose that $AE:BE=2:1$. Show that $\triangle\ ABC$ is equilateral.
G2	Consider $\triangle\ ABC$ right at B, F a point such that $B-F-C$ and AF bisects $\angle BAC$, I a point such that $A-I-F$ and CI bisect $\angle ACB$, and E a point such that $A-E-C$ and $AF\perp EI$. If $AB=4$ and $\frac{AI}{IF}=43$, determine AE. Notation: $A-B-C$ means than points A,B,C are collinear in that order i.e. B lies between A and C.
G3	Let the $JHIZ$ be a rectangle and let A and C be points on the sides ZI and ZJ, respectively. The perpendicular from A on CH intersects line HI at point X and perpendicular from C on AH intersects line HJ at point Y. Show that points X,Y, and Z are collinear.
LR1	With 21 tiles, some white and some black, a 3×7 rectangle is formed. Show that there are always four tokens of the same color located at the vertices of a rectangle.
LR2	There are 2016 participants in the Olcotournament of chess. It is known that in any set of four participants, there is one of them who faced the other three. Prove there is at least 2013 participants who faced everyone else.

LR3 Consider an arithmetic progression made up of 100 terms. If the sum of all the terms of the progression is 150 and the sum of the even terms is 50, find the sum of the squares of the 100 terms of the progression.

N1 Find all $x \in R$ such that

$$x - \left[\frac{x}{2016}\right] = 2016$$

, where $[k]$ represents the largest smallest integer or equal to k.

N2 Determine all positive integers a and b for which $a^4 + 4b^4$ be a prime number.

N3 Find all nonnegative integers a and b that satisfy the equation

$$3 \cdot 2^a + 1 = b^2.$$

– Day 2 - Shortlist

A1 Find all solutions of the system

$$\sqrt[3]{\frac{yz^4}{x^2}} + 2wx = 0 \ \sqrt[3]{\frac{xz^4}{y}} + 5wy = 0 \ \sqrt[3]{\frac{xy}{x}} + 7wz^{-1/3} = 0 \ x^{12} + \frac{125}{4}y^5 + \frac{343}{2}z^4 = 16$$

where $x, y, z \geq 0$ and $w \in R$

I attached the system, in case I have any typos

A2 The initial number of inhabitants of a city of more than 150 inhabitants is a perfect square. With an increase of 1000 inhabitants it becomes a perfect square plus a unit. After from another increase of 1000 inhabitants it is again a perfect square. Determine the quantity of inhabitants that are initially in the city.

F1 Let a, b and c be real numbers, and let $f(x) = ax^2 + bx + c$ and $g(x) = cx^2 + bx + a$ functions such that $|f(-1)| \leq 1$, $|f(0)| \leq 1$ and $|f(1)| \leq 1$. Show that if $-1 \leq x \leq 1$, then $|f(x)| \leq \frac{5}{4}$ and $|g(x)| \leq 2$.

F2 Sea $f : R^+ \to R$ defined as

$$f(x) = \frac{1}{\sqrt[3]{x^2 + 6x + 9} + \sqrt[3]{x^2 + 4x + 3} + \sqrt[3]{x^2 + 2x + 1}}$$

Calculate

$$f(1) + f(2) + f(3) + ... + f(2016).$$

F3 Let $f : Z^+ \to Z^+ \cup \{0\}$ a function that meets the following conditions:
a) $f(ab) = f(a) + f(b)$,
b) $f(a) = 0$ provided that the digits of the unit of a are 7,
c) $f(10) = 0$.
Find $f(2016)$.

G1 Let $\triangle ABC$ be acute with orthocenter H. Let X be a point on BC such that $B - X - C$. Let Γ be the circumscribed circle of $\triangle BHX$ and Γ_2 be the circumscribed circle of $\triangle CHX$. Let E be the intersection of AB with Γ , and D be the intersection of AC with Γ_2. Let L be the intersection of line HD with Γ and J be the intersection of line EH with Γ_2. Prove that points L, X, and J are collinear.

Notation: $A - B - C$ means than points A, B, C are collinear in that order i.e. B lies between A and C.

G2 Let $ABCD$ be a convex quadrilateral, such that A, B, C, and D lie on a circle, with $\angle DAB < \angle ABC$. Let I be the intersection of the bisector of $\angle ABC$ with the bisector of $\angle BAD$. Let ℓ be the parallel line to CD passing through point I. Suppose ℓ cuts segments DA and BC at L and J, respectively. Prove that $AL + JB = LJ$.

G3 Let $\triangle ABC$ be acute, with incircle Γ and incenter I. Γ touches sides AB, BC and AC at Z, X and Y, respectively. Let D be the intersection of XZ with CI and L the intersection of BI with XY. Suppose D and L are outside of $\triangle ABC$. Prove that A, D, Z, I, Y, and L lie on a circle.

N1 Let $p > 5$ be a prime such that none of its digits is divisible by 3 or 7. Prove that the equation $x^4 + p = 3y^4$ does not have integer solutions.

N2 Let x, y, z be positive integers and p a prime such that $x < y < z < p$. Also x^3, y^3, z^3 leave the same remainder when divided by p. Prove that $x + y + z$ divides $x^2 + y^2 + z^2$.

N3 Find all natural values of n and m, such that $(n - 1)2^{n-1} + 5 = m^2 + 4m$.

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