Final Round - Costa Rica 2015

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–	Level III
–	Day 1
1	Let $\triangle ABC$ be such that $\angle BAC$ is acute. The line perpendicular on side AB from C and the line perpendicular on AC from B intersect the circumscribed circle of $\triangle ABC$ at D and E respectively. If $DE = BC$, calculate $\angle BAC$.
2	In a video game, there is a board divided into squares, with 27 rows and 27 columns. The squares are painted alternately in black, gray and white as follows: • in the first row, the first square is black, the next is gray, the next is white, the next is black, and so on; • in the second row, the first is white, the next is black, the next is gray, the next is white, and so on; • in the third row, the order would be gray-white-black-gray and so on; • the fourth row is painted the same as the first, the fifth the same as the second, • the sixth the same as the third, and so on. In the box in row i and column j, there are ij coins. For example, in the box in row 15 and column 20 there are $(15)(20) = 300$ coins. Verify that in total there are, in the black squares, $9^2(13^2 + 14^2 + 15^2)$ coins.
3	Indicate (justifying your answer) if there exists a function $f : R \to R$ such that for all $x \in R$ fulfills that i) $\{f(x))\} \sin^2 x + \{x\} cos(f(x))cosx = f(x)$ ii) $f(f(x)) = f(x)$ where $\{m\}$ denotes the fractional part of m. That is, $\{2.657\} = 0.657$, and $\{-1.75\} = 0.25$.
–	Day 2
4	Find all triples (p, M, z) of integers, where p is prime, m is positive and z is negative, that satisfy the equation $$p^3 + pm + 2zm = m^2 + pz + z^2$$
5	Let $a, b \in R^+$ with $ab = 1$, prove that $$\frac{1}{a^3 + 3b} + \frac{1}{b^3 + 3a} \le \frac{1}{2}.$$

6 Given the trapezoid $ABCD$ with the $BC \parallel AD$, let C_1 and C_2 be circles with diameters AB and CD respectively. Let M and N be the intersection points of C_1 with AC and BD respectively. Let K and L be the intersection points of C_2 with AC and BD respectively. Given $M \neq A$, $N \neq B$, $K \neq C$, $L \neq D$. Prove that $NK \parallel ML$.

– Shortlist

A2 Determine, if they exist, the real values of x and y that satisfy that

$$\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} = 0$$

such that $x + y < 0$.

A3 Knowing that b is a real constant such that $b \geq 1$, determine the sum of the real solutions of the equation

$$x = \sqrt{b - \sqrt{b + x}}$$

F1 A function f defined on integers such that

$f(n) = n + 3$ if n is odd $f(n) = \frac{n}{2}$ if n is even

If k is an odd integer, determine the values for which $f(f(f(k))) = k$.

F2 Find all functions $f : R \to R$ such that $f(f(x)f(y)) = xy$ and there is no $k \in R - \{0, 1, -1\}$ such that $f(k) = k$.

G1 Points A, B, C are vertices of an equilateral triangle inscribed in a circle. Point D lies on the shorter arc $\overset{\frown}{AB}$. Prove that $AD + BD = DC$.

G3 Let $\triangle A_1B_1C_1$ and l_1, m_1, n_1 be the trisectors closest to A_1B_1, B_1C_1, C_1A_1 of the angles A_1, B_1, C_1 respectively. Let $A_2 = l_1 \cap n_1$, $B_2 = m_1 \cap l_1$, $C_2 = n_1 \cap m_1$. So on we create triangles $\triangle A_nB_nC_n$. If $\triangle A_1B_1C_1$ is equilateral prove that exists $n \in N$, such that all the sides of $\triangle A_nB_nC_n$ are parallel to the sides of $\triangle A_1B_1C_1$.

G4 Consider $\triangle ABC$, right at B, let I be its incenter and F, D, E the points where the circle inscribed on sides AB, BC and AC, respectively. If M is the intersection point of CI and EF, and N is the intersection point of DM and AB. Prove that $AN = ID$.

G5 Let A, B, C, D be points that lie on the same circle . Let F be such that the arc AF is congruent with the arc BF. Let P be the intersection point of the segments DF and AC. Let Q be intersection point of the CF and BD segments. Prove that $PQ \parallel AB$.

LR2 In the rectangle in the figure, we are going to write 12 numbers from 1 to 9, so that the sum of the four numbers written in each line is the same and the sum of the three is also equal numbers in each column. Six numbers have already been written. Determine the sum of the numbers of each row and every column.
`https://cdn.artofproblemsolving.com/attachments/7/f/3db9ded1e703c5392f258e1608a1800760d78`
`png`

LR3 Ana & Bruno decide to play a game with the following rules.:
a) Ana has cards $1, 3, 5, 7, ..., 2n - 1$
b) Bruno has cards $2, 4, 6, 8, ..., 2n$
During the first turn and all odd turns afterwards, Bruno chooses one of his cards first and reveals it to Ana, and Ana chooses one of her cards second. Whoever's card is higher gains a point. During the second turn and all even turns afterwards, Ana chooses one of her cards first and reveals it to Bruno, and Bruno chooses one of his cards second. Similarly, whoever's card is higher gains a point. During each turn, neither player can use a card they have already used on a previous turn. The game ends when all cards have been used after n turns. Determine the highest number of points Ana can earn, and how she manages to do this.

LR4 Let $P = \{(a, b)/a, b \in \{1, 2, ..., n\}, n \in N\}$ be a set of point of the Cartesian plane and draw horizontal, vertical, or diagonal segments, of length 1 or $\sqrt{2}$, so that both ends of the segment are in P and do not intersect each other. Furthermore, for each point (a, b) it is true that
i) if $a + b$ is a multiple of 3, then it is an endpoint of exactly 3 segments.
ii) if $a + b$ is an even not multiple of 3, then it is an endpoint of exactly 2 segments.
iii) if $a + b$ is an odd not multiple of 3, then it is endpoint of exactly 1 segment.
a) Check that with $n = 6$ it is possible to satisfy all the conditions.
b) Show that with $n = 2015$ it is not possible to satisfy all the conditions.

N1 Find all the values of $n \in N$ such that $n^2 = 2^n$.

N3 Find all the pairs $a, b \in N$ such that $ab - 1 | a^2 + 1$.

N4 Show that there are no triples (a, b, c) of positive integers such that
a) $a + c, b + c, a + b$ do not have common multiples in pairs.
b) $\frac{c^2}{a+b}, \frac{b^2}{a+c}, \frac{a^2}{c+b}$ are integer numbers.

– C2

– Day 1

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1 Let $ABCD$ be a quadrilateral whose diagonals are perpendicular, and let S be the intersection of those diagonals. Let K, L, M and N be the reflections of S on the sides AB, BC, CD and DA respectively. BN cuts the circumcircle of $\triangle SKN$ at E and BM cuts the circumcircle of $\triangle SLM$ at F. Prove that the quadrilateral $EFLK$ is cyclic.

2 A positive natural number n is said to be *comico* if its prime factorization is $n = p_1p_2...p_k$, with $k \geq 3$, and also the primes $p_1, ..., p_k$ they fulfill that $p_1 + p_2 = c_1^2$ $p_1 + p_2 + p_3 = c_2^2$ ... $p_1 + p_2 + ... + p_n = c_{n-1}^2$
where $c_1, c_2, ..., c_{n-1}$ are positive integers where c_1 is not divisible by 7.
Find all comico numbers less than $10,000$.

3 In a set X of n people, some know each other and others do not, where the relationship to know is symmetric; that is, if A knows B. then B knows A. On the other hand, given any4 people: A, B, C and D: if A knows B, B knows C and C knows D, then it happens at least one of the following three: A knows C, B knows D or A knows D. Prove that X can be partition into two sets Y and Z so that all elements of Y know all those of Z or no element in Y knows any in Z.

– Day 2

4 Find all triples of integers (x, y, z) not zero and relative primes in pairs such that $\frac{(y+z-x)^2}{4x}$, $\frac{(z+x-y)^2}{4y}$ and $\frac{(x+y-z)^2}{4z}$ are all integers.

5 Let $f : N^+ \to N^+$ be a function that satisfies that

$$kf(n) \leq f(kn) \leq kf(n) + k - 1, \ \forall k, n \in N^+$$

Prove that

$$f(a) + f(b) \leq f(a + b) \leq f(a) + f(b) + 1, \ \forall a, b \in N^+$$

6 Let $\triangle \ ABC$ be a triangle with circumcenter O. Let P and Q be internal points on the sides AB and AC respectively such that $\angle POB = \angle ABC$ and $\angle QOC = \angle ACB$. Show that the reflection of line BC over line PQ is tangent to the circumcircle of triangle $\triangle \ APQ$.