**Costa Rica - Final Round 2014**

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| – | level III |
| – | Day 1 |
| 1 | Consider the following figure where $AC$ is tangent to the circle of center $O$, $\angle BCD = 35^o$, $\angle BAD = 40^o$ and the measure of the minor arc $DE$ is $70^o$. Prove that points $B, O, E$ are collinear. `https://cdn.artofproblemsolving.com/attachments/4/0/fd5f8d3534d9d0676deebd696d174999c2ad7` `png` |
| 2 | Let $p_1, p_2, p_3$ be positive numbers such that $p_1 + p_2 + p_3 = 1$. If $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$ prove that $$(a_1p_1 + a_2p_2 + a_3p_3)(b_1p_1 + b_2p_2 + b_3p_3) \le (a_1b_1p_1 + a_2b_2p_2 + a_3b_3p_3)$$ |
| 3 | Find all possible pairs of integers $a$ and $b$ such that $ab = 160+90(a, b)$, where $(a, b)$ is the greatest common divisor of $a$ and $b$. |
| – | Day 2 |
| 4 | Consider the isosceles triangle $ABC$ inscribed in the semicircle of radius $r$. If the $\triangle\ BCD$ and $\triangle\ CAE$ are equilateral, determine the altitude of $\triangle\ DEC$ on the side $DE$ in terms of $r$. `https://cdn.artofproblemsolving.com/attachments/6/3/772ff9a1fd91e9fa7a7e45ef788eec7a1ba48` `png` |
| 5 | Let $f : N \to N$ such that $$f(1) = 0\ ,\ \ f(3n) = 2f(n) + 2\ ,\ \ f(3n-1) = 2f(n) + 1\ ,\ \ f(3n-2) = 2f(n).$$ Determine the smallest value of $n$ so that $f(n) = 2014.$ |
| 6 | $n$ people are in the plane, so that the closest person is unique and each one shoot this closest person with a squirt gun. If $n$ is odd, prove that there exists at least one person that nobody shot. If $n$ is even, will there always be a person who escape? Justify that. |
| – | level IV |

– Day 1

**1** Let $A$ and $B$ be the intersections of two circumferences $\Gamma_1$, and $\Gamma_2$. Let $C$ and $D$ points in $\Gamma_1$ and $\Gamma_2$ respectively such that $AC = AD$. Let $E$ and $F$ be points in $\Gamma_1$ and $\Gamma_2$, such that $\angle ABE = \angle ABF = 90^o$. Let $K_1$ and $K_2$ be circumferences with centers $E$ and $F$ and radii $EC$ and $FD$ respectively. Let $T$ be a point in the line $AB$, but outside the segment, with $T \neq A$ and $T \neq A'$, where $A'$ is the point symmetric to $A$ with respect to $B$. Let $X$ be the point of tangency of a tangent to $K_1$ passing through $T$, such that there arc two points of intersection of the line $TX$ to $K_2$. Let $Y$ and $Z$ be such points. Prove that

$$\frac{1}{XT} = \frac{1}{XY} + \frac{1}{XZ}.$$

**2** Find all positive integers $n$ such that $n! + 2$ divides $(2n)!$.

**3** There are 2014 houses in a circle. Let $A$ be one of these houses. Santa Claus enters house $A$ and leaves a gift. Then with probability $1/2$ he visits $A$'s left neighbor and with probability $1/2$ he visits $A$'s right neighbor. He leaves a gift also in that second house, and then repeats the procedure (visits with probability $1/2$ either of the neighbors, leaves a gift, etc). Santa finishes as soon as every house has received at least one gift.
Prove that any house $B$ different from $A$ has a probability of $1/2013$ of being the last house receiving a gift.

– Day 2

**4** The Olcommunity consists of the next seven people: Christopher Took, Humberto Brandybuck, German son of Isildur, Leogolas, Argimli, Samzamora and Shago Baggins. This community needs to travel from the Olcomashire to Olcomordor to save the world. Each person can take with them a total of $4$ day-provisions, that can be transferred to other people that are on the same day of traveling, as long as nobody holds more than $4$ day-provisions at the time. If a person returns to Olcomashire, they will be too tired to go out again. What is the farthest away Olcomordor can be from Olcomashire, so that Shago Baggins can get to Olcomordor, and the rest of the Olcommunity can return save to Olcomashire?

Note: All the members of the Olcommunity should eat exactly one day-provision while they are away from Olcomashire. The members only travel an integer number of days on each direction. Members of the Olcommunity may leave Olcomashire on different days.

**5** Let $ABC$ be a triangle, with $A'$, $B'$, and $C'$ the points of tangency of the incircle with $BC$, $CA$, and $AB$ respectively. Let $X$ be the intersection of the excircle with respect to $A$ with $AB$, and $M$ the midpoint of $BC$. Let $D$ be the intersection of $XM$ with $B'C'$. Show that $\angle C'A'D' = 90^o$.

 

**6** The sequences $a_n$, $b_n$ and $c_n$ are defined recursively in the following way:

$a_0 = 1/6$, $b_0 = 1/2$, $c_0 = 1/3$,

$$a_{n+1} = \frac{(a_n + b_n)(a_n + c_n)}{(a_n - b_n)(a_n - c_n)}, \ b_{n+1} = \frac{(b_n + a_n)(b_n + c_n)}{(b_n - a_n)(b_n - c_n)}, \ c_{n+1} = \frac{(c_n + a_n)(c_n + b_n)}{(c_n - a_n)(c_n - b_n)}$$

For each natural number $N$, the following polynomials are defined:

$A_n(x) = a_o + a_1x + ... + a_{2N}x^{2N}$ $B_n(x) = b_o + a_1x + ... + a_{2N}x^{2N}$ $C_n(x) = a_o + a_1x + ... + a_{2N}x^{2N}$

Assume the sequences are well defined.
Show that there is no real $c$ such that $A_N(c) = B_N(c) = C_N(c) = 0$.