National Math Olympiad (3rd Round) 2021

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–	Geometry
1	An acute triangle ABC is given. Let D be the foot of altitude dropped for A. Tangents from D to circles with diameters AB and AC intersects with the said circles at K and L, in respective. Point S in the plane is given so that $\angle ABC + \angle ABS = \angle ACB + \angle ACS = 180^\circ$. Prove that A, K, L and S lie on a circle.
2	Given an acute triangle ABC let M be the midpoint of AB. Point K is given on the other side of line AC from that of point B such that $\angle KMC = 90^\circ$ and $\angle KAC = 180^\circ - \angle ABC$. The tangent to circumcircle of triangle ABC at A intersects line CK at E. Prove that the reflection of line BC with respect to CM passes through the midpoint of line segment ME.
3	Given triangle ABC variable points X and Y are chosen on segments AB and AC, respectively. Point Z on line BC is chosen such that $ZX = ZY$. The circumcircle of XYZ cuts the line BC for the second time at T. Point P is given on line XY such that $\angle PTZ = 90^\circ$. Point Q is on the same side of line XY with A furthermore $\angle QXY = \angle ACP$ and $\angle QYX = \angle ABP$. Prove that the circumcircle of triangle QXY passes through a fixed point (as X and Y vary).
–	Combinatorics
1	Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.
2	Is it possible to arrange a permutation of Integers on the integer lattice infinite from both sides such that each row is increasing from left to right and each column increasing from up to bottom?
3	Let $n \geq 3$ be a fixed integer. There are $m \geq n+1$ beads on a circular necklace. You wish to paint the beads using n colors, such that among any $n+1$ consecutive beads every color appears at least once. Find the largest value of m for which this task is *not* possible. *Carl Schildkraut, USA*
–	Algebra

1 Positive real numbers a, b, c and d are given such that $a + b + c + d = 4$ prove that

$$\frac{ab}{a^2 - \frac{4}{3}a + \frac{4}{3}} + \frac{bc}{b^2 - \frac{4}{3}b + \frac{4}{3}} + \frac{cd}{c^2 - \frac{4}{3}c + \frac{4}{3}} + \frac{da}{d^2 - \frac{4}{3}d + \frac{4}{3}} \leq 4.$$

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2 If a, b, c and d are complex non-zero numbers such that

$$2|a - b| \leq |b|, 2|b - c| \leq |c|, 2|c - d| \leq |d|, 2|d - a| \leq |a|.$$

Prove that

$$\frac{7}{2} < \left|\frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}\right|.$$

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3 Polynomial P with non-negative real coefficients and function $f : \mathbb{R}^+ \to \mathbb{R}^+$ are given such that for all $x, y \in \mathbb{R}^+$ we have

$$f(x + P(x)f(y)) = (y + 1)f(x)$$

(a) Prove that P has degree at most 1.
(b) Find all function f and non-constant polynomials P satisfying the equality.

– Number Theory

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1 For a natural number n, $f(n)$ is defined as the number of positive integers less than n which are neither coprime to n nor a divisor of it. Prove that for each positive integer k there exist only finitely many n satisfying $f(n) = k$.

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2 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any two positive integers a and b we have

$$f^a(b) + f^b(a) \mid 2(f(ab) + b^2 - 1)$$

Where $f^n(m)$ is defined in the standard iterative manner.

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3 x_1 is a natural constant. Prove that there does not exist any natural number $m > 2500$ such that the recursive sequence $\{x_i\}_{i=1}^{\infty}$ defined by $x_{n+1} = x_n^{s(n)} + 1$ becomes eventually periodic modulo m. (That is there does not exist natural numbers N and T such that for each $n \geq N$, $m \mid x_n - x_{n+T}$).
($s(n)$ is the sum of digits of n.)

– Final Exam

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1 Is it possible to arrange natural numbers 1 to 8 on vertices of a cube such that each number divides sum of the three numbers sharing an edge with it?

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2 Given an acute triangle ABC, let AD be an altitude and H the orthocenter. Let E denote the reflection of H with respect to A. Point X is chosen on the circumcircle of triangle BDE such that $AC \| DX$ and point Y is chosen on the circumcircle of triangle CDE such that $DY \| AB$. Prove that the circumcircle of triangle AXY is tangent to that of ABC.

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3 Find all functions $f : \mathbb{Q}[x] \to \mathbb{R}$ such that:
(a) for all $P, Q \in \mathbb{Q}[x]$, $f(P \circ Q) = f(Q \circ P);$
(b) for all $P, Q \in \mathbb{Q}[x]$ with $PQ \neq 0$, $f(P \cdot Q) = f(P) + f(Q).$

($P \circ Q$ indicates $P(Q(x))$.)

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4 Arash and Babak play the following game, taking turns alternatively, on a 1400×1401 table. Arash starts and in his turns he colors k, L-corners (any three cell of a square). Babak in his turn colors one 2×2 square. Neither player is allowed to recolor any cell. Find all positive integers k for which Arash has a winning strategy.
