**Costa Rica - Final Round 2013**

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by parmenides51

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| **–** | Day 1 |
| **1** | Determine and justify all solutions $(x, y, z)$ of the system of equations: <br> $x^2 = y + z\ y^2 = x + z\ z^2 = x + y$ |
| **2** | Determine all even positive integers that can be written as the sum of odd composite positive integers. |
| **3** | Let $ABC$ be a triangle, right-angled at point $A$ and with $AB > AC$. The tangent through $A$ of the circumcircle $G$ of $ABC$ cuts $BC$ at $D$. $E$ is the reflection of $A$ over line $BC$. $X$ is the foot of the perpendicular from $A$ over $BE$. $Y$ is the midpoint of $AX$, $Z$ is the intersection of $BY$ and $G$ other than $B$, and $F$ is the intersection of $AE$ and $BC$. Prove $D, Z, F, E$ are concyclic. |
| **–** | Day 2 |
| **4** | Antonio and Beltran have impeccable logical reasoning, they put on a hat with a integer between $0$ and $19$ (including both) so that each of them sees the number that has the other (but cannot see his own number), and they must try to guess the number that have on their hat. <br> They have a timer that a bell rings every minute and the moment it rings. <br> This is when they must say if they know the number on their hat. <br> A third person tells them: "the sum of the numbers is $6$ or $11$ or $19$". At that moment it begins to run time. <br> After a minute the bell rings and neither of them says anything. The second minute passes , the doorbell rings and neither of us says anything. Time continues to pass and when the bell rings for the tenth time Antonio says that he already knows what is his number. <br> Just determine the number each has in his hat. |
| **5** | Determine the number of polynomials of degree $5$ with different coefficients in the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that they are divisible by $x^2 - x + 1$. Justify your answer. |
| **6** | Let $a$ and $b$ be positive integers (of one or more digits) such that $b$ is divisible by $a$, and if we write $a$ and $b$, one after the other in this order, we get the number $(a + b)^2$. Prove that $\frac{b}{a} = 6$. |
| **–** | Shortlist |
| **A1** | Let the real numbers $x, y, z$ be such that $x + y + z = 0$. Prove that <br> $$6(x^3 + y^3 + z^3)^2 \leq (x^2 + y^2 + z^2)^3.$$ |

 

**G2** Consider the triangle $ABC$. Let $P, Q$ inside the angle $A$ such that $\angle BAP = \angle CAQ$ and $PBQC$ is a parallelogram. Show that $\angle ABP = \angle ACP$.

**G3** Let $ABCD$ be a rectangle with center $O$ such that $\angle DAC = 60^o$. Bisector of $\angle DAC$ cuts a $DC$ at $S$, $OS$ and $AD$ intersect at $L$, $BL$ and $AC$ intersect at $M$. Prove that $SM \parallel CL$.

**F1** Find all functions $f : R \to R$ such that for all real numbers $x, y$ is satisfied that

$$f(x+y) = (f(x))^{2013} + f(y).$$

**F2** Find all functions $f : R - \{0, 2\} \to R$ that satisfy for all $x \neq 0, 2$

$$f(x) \cdot \left(f\left(\sqrt[3]{\frac{2+x}{2-x}}\right)\right)^2 = \frac{x^3}{4}$$

**LRP1** Consider a pyramid whose base is a $2013$-sided polygon. On each face of the pyramid the number $0$ is written. The following operation is carried out: a vertex is chosen from the pyramid and add or subtract $1$ from all the faces that contain that vertex. It's possible, after repeating a finite number of times the previous procedure, that all the faces of the pyramid have the number $1$ written?

**LRP2** From a set containing $6$ positive and consecutive integers they are extracted, randomly and with replacement, three numbers $a, b, c$. Determine the probability that even $a^b + c$ generates as a result .

**N1** Find all triples $(a, b, p)$ of positive integers, where $p$ is a prime number, such that $a^p - b^p = 2013$.