Stanford Mathematics Tournament 2011

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– Team Round

– *The 15 problems p1-15 come from Stanford Math Tournament. Rice Math Tournament had 13 same problems and p16, p17 as substitutes of problems p5 and p11.*

p1. Let $ABCD$ be a unit square. The point E lies on BC and F lies onAD. $\vartriangle AEF$ is equilateral. $GHIJ$ is a square inscribed in $\vartriangle AEF$ so that $\overline{GH}$ is on $\overline{EF}$. Compute the area of $GHIJ$.
`https://cdn.artofproblemsolving.com/attachments/e/1/e7c02a8c2bee27558a441e4acc9b639f084c7`
`png`

p2. Find all integers x for which $|x^3 + 6x^2 + 2x - 6|$ is prime.

p3. Let A be the set of points (a, b) with $2 < a < 6$, $-2 < b < 2$ such that the equation

$$ax^4 + 2x^3 - 2(2b - a)x^2 + 2x + a = 0$$

has at least one real root. Determine the area of A.

p4. Three nonnegative reals x, y, z satisfy $x + y + z = 12$ and $xy + yz + zx = 21$. Find the maximum of xyz.

p5. Let $\vartriangle ABC$ be equilateral. Two points D and E are on side BC (with order B, D, E, C), and satisfy $\angle DAE = 30^o$. If $BD = 2$ and $CE = 3$, what is BC?
`https://cdn.artofproblemsolving.com/attachments/a/6/4dd9249411e35efaa220b184e2dc3428493b7`
`png`

p6. Three numbers are chosen at random between 0 and 2. What is the probability that the difference between the greatest and least is less than $1/4$?

p7. Tony the mouse starts in the top left corner of a 3×3 grid. After each second, he randomly moves to an adjacent square with equal probability. What is the probability he reaches the cheese in the bottom right corner before he reaches the mousetrap in the center?

p8. Let $A = (0, 0)$, $B = (1, 0)$, and $C = (0, 1)$. Divide AB into n equal segments, and call the

endpoints of these segments $A = B_0, B_1, B_2, ..., B_n = B$. Similarly, divide AC into n equal segments with endpoints $A = C_0, C_1, C_2, ..., C_n = C$. By connecting B_i and C_{n-i} for all $0 \le i \le n$, one gets a piecewise curve consisting of the uppermost line segments. Find the equation of the limit of this piecewise curve as n goes to infinity.
`https://cdn.artofproblemsolving.com/attachments/7/d/903a6dc58d6f49be75c7aa8fecfdc863c2a74`
`png`

p9. Determine the maximum number of distinct regions into which 2011 circles of arbitrary size can partition the plane.

p10. For positive reals x, y, and z, compute the maximum possible value of $\frac{xyz(x+y+z)}{(x+y)^2(y+z)^2}$.

p11. Find the diameter of an icosahedron with side length 1 (an icosahedron is a regular polyhedron with 20 identical equilateral triangle faces, a picture is given below).
`https://cdn.artofproblemsolving.com/attachments/2/d/e5714078e9da58557cde03ae5c93644877669`
`png`

p12. Find the boundary of the projection of the sphere $x^2 + y^2 + (z - 1)^2 = 1$ onto the plane $z = 0$ with respect to the point $P = (0, -1, 2)$. Express your answer in the form $f(x, y) = 0$, where $f(x, y)$ is a function of x and y.

p13. Compute the number of pairs of 2011-tuples $(x_1, x_2, ..., x_{2011})$ and $(y_1, y_2, ..., y_{2011})$ such that $x_k = x^2_{k-1} - y^2_{k-1} - 2$ and $y_k = 2x_{k-1}y_{k-1}$ for $1 \le k \le 2010$, $x_1 = x^2_{2011} - y^2_{2011} - 2$, and $y_1 = 2x_{2011}y_{2011}$.

p14. Compute $I = \int_0^1 \frac{ln(x+1)}{x^2+1} dx$.

p15. Find the smallest $\alpha > 0$ such that there exists $m > 0$ making the following equation hold for all positive integers $a, b \ge 2$:

$$\left(\frac{1}{gcd(a, b-1)} + \frac{1}{gcd(a-1, b)}\right)(a+b)^\alpha \ge m.$$

Rice Math Tournament problems (substitutes of problems p5, and p11).

p16. If $f(x) = (x - 1)^4(x - 2)^3(x - 3)^2$, find $f'''(1) + f''(2) + f'(3)$.

p17. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1, 0, 1\}$ and with least possible degree such that $P(2010) \equiv 1$ (mod 3), $P(2011) \equiv 0$ (mod 3), and $P(2012) \equiv 0$ (mod 3).

PS. You had better use hide for answers.