2010 Spring Lexington Math Tournament

www.artofproblemsolving.com/community/c2490082

by parmenides51, GammaZero

–	Individual Round
1	Two distinct positive even integers sum to 8. Determine the larger of the 2 integers.
2	Let points A, B, and C lie on a line such that $AB = 1, BC = 1$, and $AC = 2$. Let C_1 be the circle centered at A passing through B, and let C_2 be the circle centered at A passing through C. Find the area of the region outside C_1, but inside C_2.
3	Start with a positive integer. Double it, subtract 4, halve it, then subtract the original integer to get x. What is x?
4	Determine the largest positive integer that is a divisor of all three of $A = 2^{2010} \times 3^{2010}, B = 3^{2010} \times 5^{2010}$, and $C = 5^{2010} \times 2^{2010}$.
5	Evaluate $2010^2 - 2009 \cdot 2011$.
6	Al has three red marbles and four blue marbles. He draws two different marbles at the same time. What is the probability that one is red and the other is blue?
7	Let $ABCD$ be a square with $AB = 6$. A point P in the interior is 2 units away from side BC and 3 units away from side CD. What is the distance from P to A?
8	How many members are there of the set $\{-79, -76, -73, \dots, 98, 101\}$?
9	Let ABC and BCD be equilateral triangles, such that $AB = 1$, and $A \neq D$. Find the area of triangle ABD.
10	How many integers less than 2502 are equal to the square of a prime number?
11	Compute the number of positive integers n less than 100 for which $1 + 2 + \cdots + n$ is not divisible by n.
12	Tim is thinking of a positive integer between 2 and 15, inclusive, and Ted is trying to guess the integer. Tim tells Ted how many factors his integer has, and Ted is then able to be certain of what Tim's integer is. What is Tim's integer?

13 Let ABC be a non-degenerate triangle inscribed in a circle, such that AB is the diameter of the circle. Let the angle bisectors of the angles at A and B meet at P. Determine the maximum possible value of $\angle APB$, in degrees.

---

14 On the team round, an LMT team of six students wishes to divide itself into two distinct groups of three, one group to work on part 1, and one group to work on part 2. In addition, a captain of each group is designated. In how many ways can this be done?

---

15 Let x and y be real numbers such that $x^2 + y^2 - 22x - 16y + 113 = 0$. Determine the smallest possible value of x.

---

16 Determine the number of three digit integers that are equal to 19 times the sum of its digits.

---

17 Al wishes to label the faces of his cube with the integers $2009, 2010,$ and 2011, with one integer per face, such that adjacent faces (faces that share an edge) have integers that differ by at most 1. Determine the number of distinct ways in which he can label the cube, given that two configurations that can be rotated on to each other are considered the same, and that we disregard the orientation in which each number is written on to the cube.

---

18 Let l be a line and A be a point such that A is not on l. Let P be a point on l such that segment AP and line l for a 60° angle and $AP = 1$. Extend segment AP past P to a point B on the other side of l. Then, let the perpendicular from B to l have foot M, and extend BM past M to C. Finally, extend CP past P to D. Given that $\frac{BP}{AP} = \frac{CM}{BM} = \frac{DP}{CP} = 2$, determine the are of triangle BPD.

---

19 Two integers are called *relatively prime* if they share no common factors other than 1. Determine the sum of all positive integers less than 162 that are relatively prime to 162.

---

20 Let $f(x) = x^5 - 3x^4 + 2x^3 + 6x^2 + x - 14 = a(x-1)^5 + b(x-1)^4 + c(x-1)^3 + d(x-1)^2 + e(x-1) + f$, for some real constants a, b, c, d, e, f. Determine the value of $ab + bc + cd + de + ad + be$.

– Theme Round

---

1 J has several cheetahs in his dresser, which has 7 drawers, such that each drawer has the same number of cheetahs. He notices that he can take out one drawer, and redistribute all of the cheetahs (including those in the removed drawer) in the remaining 6 drawers such that each drawer still has an equal number of cheetahs as the other drawers. If he has at least one cheetah, what is the smallest number of cheetahs that he can have?

---

2 J has 53 cheetahs in his hair, which he will put in 10 cages. Let A be the number of cheetahs in the cage with the largest number of cheetahs (there could be a tie, but in this case take the number of cheetahs in one of the cages involved in the tie). Find the least possible value of A.

---

3 J has 98 cheetahs in his pants, some of which are male and the rest of which are female. He realizes that three times the number of male cheetahs in his pants is equal to nine more than twice the number of female cheetahs. How many male cheetahs are in his pants?

---

4 Because J's cheetahs are everywhere, they are now running away. A particularly unintelligent one starts to run in a 720 mile loop at 80 miles per hour. J immediately starts to chase after it, starting at the same point, at 10 miles per hour at $12:00$ PM, but realizes one hour later that it would be more wise to turn around and run in the opposite direction in the loop, so he does this. Assuming both maintain a constant speed, at what time do J and the cheetah collide? Round to the nearest minute, and be sure to include AM or PM.

---

5 Once J and his cheetah collide, J dies a very slow and painful death. The cheetahs come back for his funeral, which is held in a circular stadium with 10 rows. The first row has 10 seats in a circle, and each subsequent row has 3 more seats. However, no two adjacent seats may be occupied due to the size of the cheetahs. What is the maximum number of cheetahs that can fit in the stadium?

---

6 Given a square $ABCD$, with $AB = 1$ mark the midpoints M and N of AB and BC, respectively. A lasar beam shot from M to N, and the beam reflects of BC, CD, DA, and comes back to M. This path encloses a smaller area inside square $ABCD$. Find this area.

---

7 Given a rectangle $EFGH$ with $EF = 3$ and $FG = 2010$, mark a point P on FG such that $FP = 4$. A laser beam is shot from E to P, which then reflects off FG, then EH, then FG, etc. Once it reaches some point of GH, the beam is absorbed; it stops reflecting. How far does the beam travel?

---

8 **Same exact problem as 2010 Spring LMT Theme Round Problem 7?** :what?:

---

9 Given a triangle XYZ with $\angle Y = 90^\circ, XY = 1$, and $XZ = 2$, mark a point Q on YZ such that $\frac{ZQ}{ZY} = \frac{1}{3}$. A laser beam is shot from Q perpendicular to YZ, and it reflects off the sides of XYZ indefinitely. How many bounces does it take for the laser beam to get back to Q for the first time (not including the release from Q and the return to Q)?

---

10 Given a triangle XYZ with $\angle Y = 90^\circ, XY = 1$, and $XZ = 2$, mark a point Q on YZ such that $\frac{ZQ}{ZY} = \frac{1}{3}$. A laser beam is shot from Q perpendicular to YZ, and it reflects off the sides of XYZ indefinitely. How far has the laser traveled when it reaches its 2010th bounce?

---

11 Al and Bob are playing Rock Paper Scissors. Al plays rock. What is the probability that Al wins, given that Bob plays randomly and has an equal probability of playing rock, paper, and scissors?

---

12 Al and Bob play Rock Paper Scissors until someone wins a game. What is the probability that this happens on the sixth game?

---

14 Al and Bob are joined by Carl and D'Angelo, and they decide to play a team game of Rock Paper Scissors. A game is called *perfect* if some two of the four play the same thing, and the other two also play the same thing, but something different. For example, an example of a perfect game would be Al and Bob playing rock, and Carl and D'Angelo playing scissors, but if all four play paper, we do not have a perfect game. What is the probability of a perfect game?

---

15 Al is bored of Rock Paper Scissors, and wants to invent a new game: $Z-Y-X-W-V$. Two players, each choose to play either $Z, Y, X, W,$ or V. If they play the same thing, the result is a tie. However, Al must come up with a 'pecking order', that is, he must decide which plays beat which. For each of the 10 pairs of distinct plays that the two players can make, Al randomly decides a winner. For example, he could decide that W beats Y and that Z beats X, etc. What is the probability that after Al makes all of these 10 choices, the game is balanced, that is, playing each letter results in an equal probability of winning?

– Guts Round

---

1 Compute $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$.

---

2 If you increase a number X by $20\%,$ you get Y. By what percent must you decrease Y to get X?

---

3 A circle has circumference 8π. Determine its radius.

---

4 The perimeter of a square is equal in value to its area. Determine the length of one of its sides.

---

5 Big Welk writes the letters of the alphabet in order, and starts again at A each time he gets to Z. What is the 4^3-rd letter that he writes down?

---

6 Al travels for 20 miles per hour rolling down a hill in his chair for two hours, then four miles per hour climbing a hill for six hours. What is his average speed, in miles per hour?

---

7 A team of four students goes to LMT, and each student brings a lunch. However, on the bus, the students' lunches get mixed up, and during lunch time, each student chooses a random lunch to eat (no two students may eat the same lunch). What is the probability that each student chooses his or her own lunch correctly?

---

8 The integer 111111 is the product of five prime numbers. Determine the sum of these prime numbers.

---

9 A trapezoid has bases with lengths equal to 5 and 15 and legs with lengths equal to 13 and 13. Determine the area of the trapezoid.

---

10 A two digit prime number is such that the sum of its digits is 13. Determine the integer.

---

11 Carl, James, Saif, and Ted play several games of two-player For The Win on the Art of Problem Solving website. If, among these games, Carl wins 5 and loses 0, James wins 4 and loses 2, Saif wins 1 and loses 6, and Ted wins 4, how many games does Ted lose?

---

12 a, b, c, d, e are equal to $1, 2, 3, 4, 5$ in some order, such that no two of a, b, c, d, e are equal to the same integer. Given that $b \leq d, c \geq a, a \leq e, b \geq e$, and that $d \neq 5$, determine the value of $a^b + c^d + e$.

---

13 A circle with center O has radius 5, and has two points A, B on the circle such that $\angle AOB = 90^\circ$. Rays OA and OB are extended to points C and D, respectively, such that AB is parallel to CD, and the length of CD is 200% more than the radius of circle O. Determine the length of AC.

---

14 Seongcheol has 3 red shirts and 2 green shirts, such that he cannot tell the difference between his three red shirts and he similarly cannot tell the difference between his two green shirts. In how many ways can he hang them in a row in his closet, given that he does not want the two green shirts next to each other?

---

15 Determine the number of ordered pairs (x, y) with x and y integers between -5 and 5, inclusive, such that $(x + y)(x + 3y) = (x + 2y)^2$.

---

16 Al has three bags, each with three marbles each. Bag 1 has two blue marbles and one red marble, Bag 2 has one blue marble and two red marbles, and Bag 3 has three red marbles. He chooses two distinct bags at random, then one marble at random from each of the chosen bags. What is the probability that he chooses two blue marbles?

---

17 Determine the sum of the two largest prime factors of the integer $89! + 90!$.

---

18 Congruent unit circles intersect in such a way that the center of each circle lies on the circumference of the other. Let R be the region in which two circles overlap. Determine the perimeter of R.

---

19 Let $f(x) = x^2 - 2x + 1$. For some constant k, $f(x + k) = x^2 + 2x + 1$ for all real numbers x. Determine the value of k.

---

20 Three vertices of a parallelogram are $(2, -4)$, $(-2, 8)$, and $(12, 7.)$ Determine the sum of the three possible x-coordinates of the fourth vertex.

---

21 Jae and Yoon are playing SunCraft. The probability that Jae wins the n-th game is $\frac{1}{n+2}$. What is the probability that Yoon wins the first six games, assuming there are no ties?

---

22 Two circles, ω_1 and ω_2, intersect at X and Y. The segment between their centers intersects ω_1 and ω_2 at A and B respectively, such that $AB = 2$. Given that the radii of ω_1 and ω_2 are 3 and 4, respectively, find XY.

23 In how many ways can six marbles be placed in the squares of a 6-by-6 grid such that no two marbles lie in the same row or column?

24 Let ABC be an equilateral triangle with $AB = 1$. Let M be the midpoint of BC, and let P be on segment AM such that $AM/MP = 4$. Find BP.

25-27 25. Let C be the answer to Problem 27. What is the C-th smallest positive integer with exactly four positive factors?

26. Let A be the answer to Problem 25. Determine the absolute value of the difference between the two positive integer roots of the quadratic equation $x^2 - Ax + 48 = 0$
27. Let B be the answer to Problem 26. Compute the smallest integer greater than $\frac{B}{\pi}$

28 Two knights placed on distinct square of an 8×8 chessboard, whose squares are unit squares, are said to attack each other if the distance between the centers of the squares on which the knights lie is $\sqrt{5}$. In how many ways can two identical knights be placed on distinct squares of an 8×8 chessboard such that they do NOT attack each other?

29 Let S be the set of integers that represent the number of intersections of some four distinct lines in the plane. List the elements of S in ascending order.

30 Rick has 7 books on his shelf: three identical red books, two identical blue books, a yellow book, and a green book. Dave accidentally knocks over the shelf and has to put the books back on in the same order. He knows that none of the red books were next to each other and that the yellow book was one of the first four books on the shelf, counting from the left. If Dave puts back the books according to the rules, but otherwise randomly, what is the probability that he puts the books back correctly?

31 In how many ways can each of the integers 1 through 11 be assigned one of the letters L, M, and T such that consecutive multiples of n, for any positive integer n, are not assigned the same letter?

32 Compute the infinite sum $\frac{1^3}{2^1} + \frac{2^3}{2^2} + \frac{3^3}{2^3} + \cdots + \frac{n^3}{2^n} + \ldots.$

33 Let $ABCD$ be a unit square. E and F trisect AB such that $AE < AF.G$ and H trisect BC such that $BG < BH.I$ and J bisect CD and DA, respectively. Let HJ and EI meet at K, and let GJ and FI meet at L. Compute the length KL.

34 A *prime power* is an integer of the form p^k, where p is a prime and k is a nonnegative integer. How many prime powers are there less than or equal to 10^6? Your score will be $16 - 80|\frac{\text{Your Answer}}{\text{Actual Answer}} - 1|$ rounded to the nearest integer or 0, whichever is higher.

35 Consider a set of 6 fixed points in the plane, with no three collinear. Between some pairs of these points, we may draw one arrow from one point to the other. How many possible configurations of arrows are there such that if there is an arrow from point A to point B and an arrow from B to C, then there is an arrow from A to C? Your score will be $16 - \frac{1}{800}|$**Your Answer** $-$ **Actual Answer**$|$ rounded to the nearest integer or zero, whichever is higher.

---

36 Write down one of the following integers: $1, 2, 4, 8, 16$. If your team is the only one that submits this integer, you will receive that number of points; otherwise, you receive zero.

There's no real way to solve this but during the competition, each of the 5 available scores were submitted at least twice by the 16 teams competing.

– Team Round

Team Round **p1.** I open my 2010-page dictionary, whose pages are numbered 1 to 2010 starting on page 1 on the right side of the spine when opened, and ending with page 2010 on the left. If I open to a random page, what is the probability that the two page numbers showing sum to a multiple of 6?

p2. Let A be the number of positive integer factors of 128.
Let B be the sum of the distinct prime factors of 135.
Let C be the units' digit of 381.
Let D be the number of zeroes at the end of $2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^1$.
Let E be the largest prime factor of 999.
Compute $\sqrt[3]{\sqrt{A+B}+\sqrt[3]{D^C+E}}$.

p3. The root mean square of a set of real numbers is defined to be the square root of the average of the squares of the numbers in the set. Determine the root mean square of 17 and 7.

p4. A regular hexagon $ABCDEF$ has area 1. The sidesAB, CD, and EF are extended to form a larger polygon with $ABCDEF$ in the interior. Find the area of this larger polygon.

p5. For real numbers x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 3 \rfloor = 3$ and $\lfloor 5.2 \rfloor = 5$. Evaluate $\lfloor -2.5 \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor -\sqrt{2} \rfloor + \lfloor 2.5 \rfloor$.

p6. The mean of five positive integers is 7, the median is 8, and the unique mode is 9. How many possible sets of integers could this describe?

p7. How many three digit numbers x are there such that $x + 1$ is divisible by 11?

p8. Rectangle $ABCD$ is such that $AD = 10$ and $AB > 10$. Semicircles are drawn with diameters AD and BC such that the semicircles lie completely inside rectangle $ABCD$. If the area of the region inside $ABCD$ but outside both semicircles is 100, determine the shortest possible distance between a point X on semicircle AD and Y on semicircle BC.

p9. 8 distinct points are in the plane such that five of them lie on a line ℓ, and the other three points lie off the line, in a way such that if some three of the eight points lie on a line, they lie on ℓ. How many triangles can be formed using some three of the 8 points?

p10. Carl has 10 Art of Problem Solving books, all exactly the same size, but only 9 spaces in his bookshelf. At the beginning, there are 9 books in his bookshelf, ordered in the following way.
$A - B - C - D - E - F - G - H - I$
He is holding the tenth book, J, in his hand. He takes the books out one-by-one, replacing each with the book currently in his hand. For example, he could take out A, put J in its place, then take out D, put A in its place, etc. He never takes the same book out twice, and stops once he has taken out the tenth book, which is G. At the end, he is holding G in his hand, and his bookshelf looks like this. $C - I - H - J - F - B - E - D - A$
Give the order (start to finish) in which Carl took out the books, expressed as a 9-letter string (word).

PS. You had better use hide for answers.

– Theme Round

1 Cheetahs

– Cheetahs in my dresser,
Cheetahs in my hair.
Cheetahs in my pants,
Cheetahs everywhere!
-A poem by J. Samuel Trabucco, Esq.

p1. J has several cheetahs in his dresser, which has 7 drawers, such that each drawer has the same number of cheetahs. He notices that he can take out one drawer, and redistribute all of the cheetahs (including those in the removed drawer) in the remaining 6 drawers such that each drawer still has an equal number of cheetahs as the other drawers. If he has at least one cheetah, what is the smallest number of cheetahs that he can have?

p2. J has 53 cheetahs in his hair, which he will put in 10 cages. Let A be the number of cheetahs in the cage with the largest number of cheetahs (there could be a tie, but in this case take the number of cheetahs in one of the cages involved in the tie). Find the least possible value of A.

p3. J has **98** cheetahs in his pants, some of which are male and the rest of which are female. He realizes that three times the number of male cheetahs in his pants is equal to nine more than twice the number of female cheetahs. How many male cheetahs are in his pants?

p4. Because J's cheetahs are everywhere, they are now running away. A particularly unintelligent one starts to run in a 720 mile loop at 80 miles per hour. J immediately starts to chase after it, starting at the same point, at 10 miles per hour at $12:00$ PM, but realizes one hour later that it would be more wise to turn around and run in the opposite direction in the loop, so he does this. Assuming both maintain a constant speed, at what time do J and the cheetah collide? Round to the nearest minute, and be sure to include AM or PM.

p5. Once J and his cheetah collide, J dies a very slow and painful death. The cheetahs come back for his funeral, which is held in a circular stadium with 10 rows. The first row has 10 seats in a circle, and each subsequent row has 3 more seats. However, no two adjacent seats may be occupied due to the size of the cheetahs. What is the maximum number of cheetahs that can fit in the stadium?

PS. You should use hide for answers.

2 Lasers

– Laser beams are known for reflecting off solid objects. Whenever a laser beam hits a straight, solid wall, it reflects off in the opposite direction, at an angle to the wall that is equal to the angle at which it hits, as shown.
https://cdn.artofproblemsolving.com/attachments/8/2/e56b311d1cea61b999d36cbe9189b84586a7(
png

The path of the laser L meets the wall, W, at an angle of 55.15 degrees coming from the right. Then, L reflects off W to a redirected path, L' , at an angle of 55.15 degrees going off to the left. In particular, if L were to meet W at an angle of 90 degrees, then L' would follow L, going backwards (straight up).

p6. Given a square $ABCD$, with $AB = 1$, mark the midpoints M and N of AB and BC, respectively. A laser beam shot from M to N, and the beam reflects of BC, CD, DA, and comes back to M. This path encloses a smaller area inside square $ABCD$. Find this area.

p7. Given a rectangle $EFGH$, with $EF = 3$ and $FG = 40$, mark a point P on FG such that $FP = 4$. A laser beam is shot from E to P, which then reflects off FG, then EH, then FG, etc. Once it reaches some point on GH, the beam is absorbed; it stops reflecting. How far does the beam travel?

p8. Given a rectangle $EFGH$ with $EF = 3$ and $FG = 2010$, mark a point P on FG such that $FP = 4$. A laser beam is shot from E to P, which then reflects off FG, then EH, then FG, etc. Once it reaches some point on GH, the beam is absorbed; it stops reflecting. How far does the beam travel?

p9. Given a triangle XYZ with $\angle Y = 90^o$, $XY = 1$, and $XZ = 2$, mark a point Q on YZ such that $\frac{ZQ}{ZY} = \frac{1}{3}$. A laser beam is shot from Q perpendicular to YZ, and it reflects off the sides of XYZ indefinitely. How many bounces does it take for the laser beam to get back to Q for the first time (not including the release from Q and the return to Q)?

p10. Given a triangle XYZ with $\angle Y = 90^o$, $XY = 1$, and $XZ = 2$, mark a point Q on YZ such that $\frac{ZQ}{ZY} = \frac{1}{3}$. A laser beam is shot from Q perpendicular to YZ, and it reflects off the sides of XYZ indefinitely. How far has the laser traveled when it reaches its 2010th bounce?

PS. You should use hide for answers.

---

3 Rock Paper Scissors

– In the game of Rock Paper Scissors, typically two opponents make, with their hand, either a rock, a piece of paper, or a pair of scissors. If the two players play rock and scissors, the one who plays rock wins. If they play scissors and paper, the one who plays scissors wins, and if they play paper and rock, the one who plays paper wins. If they play the same thing, the result is a tie.

p11. Al and Bob are playing Rock Paper Scissors. Al plays rock. What is the probability that Al wins, given that Bob plays randomly and has an equal probability of playing rock, paper, and scissors?

p12. In a given game, what is the probability of a tie, given that both players play randomly and with an equal probability of playing rock, paper, and scissors?

p13. Al and Bob play Rock Paper Scissors until someone wins a game. What is the probability that this happens on the sixth game?

p14. Al and Bob are joined by Carl and D'Angelo, and they decide to play a team game of Rock Paper Scissors. A game is called perfect if some two of the four play the same thing, and the other two also play the same thing, but something different. For example, an example of a perfect game would be Al and Bob playing rock, and Carl and D'Angelo playing scissors, but if all four play paper, we do not have a perfect game. What is the probability of a perfect game?

p15. Al is bored of Rock Paper Scissors, and wants to invent a new game: $Z-Y-X-W-V$. Two players, each choose to play either Z, Y, X, W, or V. If they play the same thing, the result is a tie. However, Al must come up with a 'pecking order', that is, he must decide which plays beat which. For each of the 10 pairs of distinct plays that the two players can make, Al randomly decides a winner. For example, he could decide that W beats Y and that Z beats X, etc. What is the probability that after Al makes all of these 10 choices, the game is balanced, that is, playing each letter results in an equal probability of winning?

PS. You should use hide for answers.