Berkeley Math Tournament , 2012 Spring , Individual + Team + Tournament Round
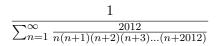
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by parmenides51, mathisawesome

– Individual Round

1 Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic progression with $a_1 > 0$ and $5 \cdot a_{13} = 6 \cdot a_{19}$. What is the smallest integer n such that $a_n < 0$?

2 Find the smallest number with exactly 28 divisors.

3 Find the largest prime factor of

$$\frac{1}{\sum_{n=1}^{\infty} \frac{2012}{n(n+1)(n+2)(n+3)...(n+2012)}}$$

4 Tyler rolls two 4025 sided fair dice with sides numbered $1, \ldots, 4025$. Given that the number on the first die is greater than or equal to the number on the second die, what is the probability that the number on the first die is less than or equal to 2012?

5 Let $a \uparrow\uparrow b = a^{a^{a \cdots a^a}}$, where there are b a's in total. That is $a \uparrow\uparrow b$ is given by the recurrence

$$a \uparrow\uparrow b = \begin{cases} a & b = 1 \\ a^{a \uparrow\uparrow (b-1)} & b \geq 2 \end{cases}$$

What is the remainder of $3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3))$ when divided by 60?

6 Let ABCD be a cyclic quadrilateral, with AB $= 7$, BC $= 11$, CD $= 13$, and DA $= 17$. Let the incircle of ABD hit BD at R and the incircle of CBD hit BD at S. What is RS?

7 Let a , b , c , d , $(a + b + c + 18 + d)$, $(a + b + c + 18 - d)$, $(b + c)$, and $(c + d)$ be distinct prime numbers such that $a + b + c = 2010, a, b, c, d \neq 3$, and $d \leq 50$. Find the maximum value of the difference between two of these prime numbers.

8 You are tossing an unbiased coin. The last 28 consecutive flips have all resulted in heads. Let x be the expected number of additional tosses you must make before you get 60 consecutive heads. Find the sum of all distinct prime factors in x.

9 Bowling Pins is a game played between two players in the following way:
We start with 14 bowling pins in a line:

$X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X$

Players alternate turns. On each turn, the player can knock down one, two or three consecutive pins
at a time. For example:

Jing Jing bowls:

$X \quad X$ $\qquad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X$

Soumya bowls:

$X \quad X$ $\qquad X \quad X \quad X$ $\qquad X \quad X \quad X \quad X \quad X \quad X$

Jing Jing bowls again:

$X \quad X$ $\qquad X \quad X \quad X$ $\qquad X \quad X$ $\qquad X$

The player who knocks down the last pin wins.
In the above game, it is Soumya's turn. If he plays perfectly from here, he has a winning strategy (In
fact, he has four different winning moves.)
Imagine it's Jing Jing's turn to play and the game looks as follows

X $\qquad X \ldots$

with 1 X on the left and a string of k consecutive X's on the right.
For what values of k from 1 to 10 does she have a winning strategy?

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10   You are at one vertex of a equilateral triangle with side length 1. All of the edges of the equilateral triangle will reflect the laser beam perfectly (angle of incidence is equal to angle of reflection). Given that the laser beam bounces off exactly 137 edges and returns to the original vertex without touching any other vertices, let M be the maximum possible distance the beam could have traveled, and m be the minimum possible distance the beam could have traveled. Find $M^2 - m^2$.

–   Team Round

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1   Let S be the set of all rational numbers $x \in [0, 1]$ with repeating base 6 expansion

$$x = 0.\overline{a_1 a_2 \ldots a_k} = 0.a_1 a_2 \ldots a_k a_1 a_2 \ldots a_k \ldots$$

for some finite sequence $\{a_i\}_{i=1}^k$ of distinct nonnegative integers less than 6. What is the sum of all numbers that can be written in this form? (Put your answer in base 10.)

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2   Evaluate $\prod_{k=1}^{254} \log_{k+1}(k+2)^{u_k}$, where $u_k = \begin{cases} -k & \text{if } k \text{ is odd} \\ \frac{1}{k-1} & \text{if } k \text{ is even} \end{cases}$

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3 Let ABC be a triangle with side lengths $AB = 2011$, $BC = 2012$, $AC = 2013$. Create squares $S_1 = ABB'A''$, $S_2 = ACC''A'$, and $S_3 = CBB''C'$ using the sides AB, AC, BC respectively, so that the side $B'A''$ is on the opposite side of AB from C, and so forth. Let square S_4 have side length $A''A'$, square S_5 have side length $C''C'$, and square S_6 have side length $B''B'$. Let $A(S_i)$ be the area of square S_i. Compute $\frac{A(S_4)+A(S_5)+A(S_6)}{A(S_1)+A(S_2)+A(S_3)}$?

4 There are 12 people labeled $1, ..., 12$ working together on 12 missions, with people $1, ..., i$ working on the ith mission. There is exactly one spy among them. If the spy is not working on a mission, it will be a huge success, but if the spy is working on the mission, it will fail with probability $1/2$. Given that the first 11 missions succeed, and the 12th mission fails, what is the probability that person 12 is the spy?

5 Let $p > 1$ be relatively prime to 10. Let n be any positive number and d be the last digit of n. Define $f(n) = \lfloor \frac{n}{10} \rfloor + d \cdot m$. Then, we can call m a *divisibility multiplier* for p, if $f(n)$ is divisible by p if and only if n is divisible by p. Find a divisibility multiplier for 2013.

6 A circle with diameter AB is drawn, and the point P is chosen on segment AB so that $\frac{AP}{AB} = \frac{1}{42}$. Two new circles a and b are drawn with diameters AP and PB respectively. The perpendicular line to AB passing through P intersects the circle twice at points S and T. Two more circles s and t are drawn with diameters SP and ST respectively. For any circle ω let $A(\omega)$ denote the area of the circle. What is $\frac{A(s)+A(t)}{A(a)+A(b)}$?

7 Suppose Bob begins walking at a constant speed from point N to point S along the path indicated by the following figure.
https://cdn.artofproblemsolving.com/attachments/6/2/f5819267020f2bd38e52c6e873a2cf91ce8c4
png
After Bob has walked a distance of x, Alice begins walking at point N, heading towards point S along the same path. Alice walks 1.28 times as fast as Bob when they are on the same line segment and 1.06 times as fast as Bob otherwise. For what value of x do Alice and Bob meet at point S?

8 Let ϕ be the Euler totient function. Let $\phi^k(n) = (\underbrace{\phi \circ ... \circ \phi}_{k})(n)$ be ϕ composed with itself k times.

Define $\theta(n) = min\{k \in N | \phi^k(n) = 1\}$
. For example, $\phi^1(13) = \phi(13) = 12$ $\phi^2(13) = \phi(\phi(13)) = 4$ $\phi^3(13) = \phi(\phi(\phi(13))) = 2$ $\phi^4(13) = \phi(\phi(\phi(\phi(13)))) = 1$
so $\theta(13) = 4$. Let $f(r) = \theta(13^r)$. Determine $f(2012)$.

9 A permutation of a set is a bijection from the set to itself. For example, if σ is the permutation $17 \mapsto 3$, $2 \mapsto 1$, and $3 \mapsto 2$, and we apply it to the ordered triplet $(1, 2, 3)$, we get the reordered

triplet $(3, 1, 2)$. Let σ be a permutation of the set $\{1, ..., n\}$. Let

$$\theta_k(m) = \begin{cases} m+1 & \text{for } m < k \\ 1 & \text{for } m = k \\ m & \text{for } m > k \end{cases}$$

Call a finite sequence $\{a_i\}_{i=1}^j$ a disentanglement of σ if $\theta_{a_j} \circ ... \circ \theta_{a_j} \circ \sigma$ is the identity permutation. For example, when $\sigma = (3, 2, 1)$, then $\{2, 3\}$ is a disentaglement of σ. Let $f(\sigma)$ denote the minimum number k such that there is a disentanglement of σ of length k. Let $g(n)$ be the expected value for $f(\sigma)$ if σ is a random permutation of $\{1, ..., n\}$. What is $g(6)$?

10 Suppose that 728 coins are set on a table, all facing heads up at first. For each iteration, we randomly choose 314 coins and flip them (from heads to tails or vice versa). Let a/b be the expected number of heads after we finish 4001 iterations, where a and b are relatively prime. Find $a + b$ mod 10000.

– Tournament Round

round 1 p1. Find all prime factors of 8051.

p2. Simplify

$$[\log_{xyz}(x^z)][1 + \log_x y + \log_x z],$$

where $x = 628$, $y = 233$, $z = 340$.

p3. In prokaryotes, translation of mRNA messages into proteins is most often initiated at start codons on the mRNA having the sequence AUG. Assume that the mRNA is single-stranded and consists of a sequence of bases, each described by a single letter A,C,U, or G.
Consider the set of all pieces of bacterial mRNA six bases in length. How many such mRNA sequences have either no A's or no U's?

p4. What is the smallest positive n so that $17^n + n$ is divisible by 29?

p5. The legs of the right triangle shown below have length $a = 255$ and $b = 32$. Find the area of the smaller rectangle (the one labeled R).
https://cdn.artofproblemsolving.com/attachments/c/d/566f2ce631187684622dfb43f36c7e759e2f3
png

p6. A 3 dimensional cube contains "cubes" of smaller dimensions, ie: faces (2-cubes),edges (1-cubes), and vertices (0-cubes). How many 3-cubes are in a 5-cube?

PS. You had better use hide for answers.

round 2   p1. 4 balls are distributed uniformly at random among 6 bins. What is the expected number of empty bins?

p2. Compute $\binom{150}{20}$ (mod 221).

p3. On the right triangle ABC, with right angle at B, the altitude BD is drawn. E is drawn on BC such that AE bisects angle BAC and F is drawn on AC such that BF bisects angle CBD. Let the intersection of AE and BF be G. Given that $AB = 15$,$BC = 20$, $AC = 25$, find $\frac{BG}{GF}$.

p4. What is the largest integer n so that $\frac{n^2-2012}{n+7}$ is also an integer?

p5. What is the side length of the largest equilateral triangle that can be inscribed in a regular pentagon with side length 1?

p6. Inside a LilacBall, you can find one of 7 different notes, each equally likely. Delcatty must collect all 7 notes in order to restore harmony and save Kanto from eternal darkness. What is the expected number of LilacBalls she must open in order to do so?

PS. You had better use hide for answers.

round 3   p1. Let $A(S)$ denote the average value of a set S. Let T be the set of all subsets of the set $\{1, 2, 3, 4, ..., 2012\}$, and let R be $\{A(K)|K \in T\}$. Compute $A(R)$.

p2. Consider the minute and hour hands of the Campanile, our clock tower. During one single day ($12 : 00$ AM - $12 : 00$ AM), how many times will the minute and hour hands form a right-angle at the center of the clock face?

p3. In a regular deck of 52 face-down cards, Billy flips 18 face-up and shuffles them back into the deck. Before giving the deck to Penny, Billy tells her how many cards he has flipped over, and blindfolds her so she can't see the deck and determine where the face-up cards are. Once Penny is given the deck, it is her job to split the deck into two piles so that both piles contain the same number of face-up cards. Assuming that she knows how to do this, how many cards should be in each pile when he is done?

p4. The roots of the equation $x^3 + ax^2 + bx + c = 0$ are three consecutive integers. Find the

maximum value of $\frac{a^2}{b+1}$.

p5. Oski has a bag initially filled with one blue ball and one gold ball. He plays the following game: first, he removes a ball from the bag. If the ball is blue, he will put another blue ball in the bag with probability $\frac{1}{437}$ and a gold ball in the bag the rest of the time. If the ball is gold, he will put another gold ball in the bag with probability $\frac{1}{437}$ and a blue ball in the bag the rest of the time. In both cases, he will put the ball he drew back into the bag. Calculate the expected number of blue balls after 525600 iterations of this game.

p6. Circles A and B intersect at points C and D. Line AC and circle B meet at E, line BD and circle A meet at F, and lines EF and AB meet at G. If $AB = 10$, $EF = 4$, $FG = 8$, find BG.

PS. You had better use hide for answers.

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round 4   p1. Denote $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$. What is $144169 \cdot S_{144169} - (S_1 + S_2 + ... + S_{144168})$?

p2. Let A, B, C be three collinear points, with $AB = 4$, $BC = 8$, and $AC = 12$. Draw circles with diameters AB, BC, and AC. Find the radius of the two identical circles that will lie tangent to all three circles.

p3. Let $s(i)$ denote the number of 1's in the binary representation of i. What is

$$\sum_{x=1} 314 \left(\sum_{i=0}^{2^{576}-2} x^{s(i)} \right) \ mod \ 629?$$

p4. Parallelogram $ABCD$ has an area of S. Let $k = 42$. E is drawn on AB such that $AE = \frac{AB}{k}$. F is drawn on CD such that $CF = \frac{CD}{k}$. G is drawn on BC such that $BG = \frac{BC}{k}$. H is drawn on AD such that $DH = \frac{AD}{k}$. Line CE intersects BH at M, and DG at N. Line AF intersects DG at P, and BH at Q. If S_1 is the area of quadrilateral $MNPQ$, find $\frac{S_1}{S}$.

p5. Let ϕ be the Euler totient function. What is the sum of all n for which $\frac{n}{\phi(n)}$ is maximal for $1 \le n \le 500$?

p6. Link starts at the top left corner of an 12×12 grid and wants to reach the bottom right corner. He can only move down or right. A turn is defined a down move immediately followed by a right move, or a right move immediately followed by a down move. Given that he makes exactly 6 turns, in how many ways can he reach his destination?

PS. You had better use hide for answers.

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round 5   p1. Let n be the number so that $1 - 2 + 3 - 4 + ... - (n-1) + n = 2012$. What is $4^{2012} \pmod{n}$?

p2. Consider three unit squares placed side by side. Label the top left vertex P and the bottom four vertices A, B, C, D respectively. Find $\angle PBA + \angle PCA + \angle PDA$.

p3. Given $f(x) = \frac{3}{x-1}$, then express $\frac{9(x^2-2x+1)}{x^2-8x+16}$ entirely in terms of $f(x)$. In other words, x should not be in
your answer, only $f(x)$.

p4. Right triangle with right angle B and integer side lengths has BD as the altitude. E and F are the incenters of triangles ADB and BDC respectively. Line EF is extended and intersects BC at G, and AB at H. If $AB = 15$ and $BC = 8$, find the area of triangle BGH.

p5. Let $a_1, a_2, ..., a_n$ be a sequence of real numbers. Call a k-inversion $(0 < k \leq n)$ of a sequence to be indices $i_1, i_2, .., i_k$ such that $i_1 < i_2 < .. < i_k$ but $a_{i1} > a_{i2} > ... > a_{ik}$. Calculate the expected number of 6-inversions in a random permutation of the set $\{1, 2, ..., 10\}$.

p6. Chell is given a strip of squares labeled $1, .., 6$ all placed side to side. For each $k \in 1, ..., 6$, she then chooses one square at random in $\{1, ..., k\}$ and places a Weighted Storage Cube there. After she has placed all 6 cubes, she computes her score as follows: For each square, she takes the number of cubes in the pile and then takes the square (i.e. if there were 3 cubes in a square, her score for that square would be 9). Her overall score is the sum of the scores of each square. What is the expected value of her score?

PS. You had better use hide for answers.

Championship p1. If n is a positive integer such that $2n + 1 = 144169^2$, find two consecutive numbers whose squares add up to $n + 1$.

p2. Katniss has an n-sided fair die which she rolls. If $n > 2$, she can either choose to let the value rolled be her score, or she can choose to roll a $n - 1$ sided fair die, continuing the process. What is the expected value of her score assuming Katniss starts with a 6 sided die and plays to maximize this expected value?

p3. Suppose that $f(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, and that $f(1) = f(2) = f(3) = f(4) = f(5) = f(6) = 7$. What is a?

p4. a and b are positive integers so that $20a + 12b$ and $20b - 12a$ are both powers of 2, but $a + b$ is not. Find the minimum possible value of $a + b$.

p5. Square $ABCD$ and rhombus $CDEF$ share a side. If $m\angle DCF = 36^o$, find the measure of $\angle AEC$.

p6. Tom challenges Harry to a game. Tom first blindfolds Harry and begins to set up the game. Tom places 4 quarters on an index card, one on each corner of the card. It is Harry's job to flip all the coins either face-up or face-down using the following rules:
(a) Harry is allowed to flip as many coins as he wants during his turn.
(b) A turn consists of Harry flipping as many coins as he wants (while blindfolded). When he is happy with what he has flipped, Harry will ask Tom whether or not he was successful in flipping all the coins face-up or face-down. If yes, then Harry wins. If no, then Tom will take the index card back, rotate the card however he wants, and return it back to Harry, thus starting Harry's next turn. Note that Tom cannot touch the coins after he initially places them before starting the game.
Assuming that Tom's initial configuration of the coins weren't all face-up or face-down, and assuming that Harry uses the most efficient algorithm, how many moves maximum will Harry need in order to win? Or will he never win?

PS. You had better use hide for answers.

Consolation p1. How many ways can we arrange the elements $\{1, 2, ..., n\}$ to a sequence $a_1, a_2, ..., a_n$ such that there is only exactly one a_i, a_{i+1} such that $a_i > a_{i+1}$?

p2. How many distinct (non-congruent) triangles are there with integer side-lengths and perimeter 2012?

p3. Let ϕ be the Euler totient function, and let $S = \{x | \frac{x}{\phi(x)} = 3\}$. What is $\sum_{x \in S} \frac{1}{x}$?

p4. Denote $f(N)$ as the largest odd divisor of N. Compute $f(1) + f(2) + f(3) + ... + f(29) + f(30)$.

p5. Triangle ABC has base AC equal to 218 and altitude 100. Squares $s_1, s_2, s_3, ...$ are drawn such that s_1 has a side on AC and has one point each touching AB and BC, and square s_k has a side on square s_{k-1} and also touches AB and BC exactly once each. What is the sum of the area of these squares?

p6. Let P be a parabola $6x^2 - 28x + 10$, and F be the focus. A line ℓ passes through F and intersects the parabola twice at points $P_1 = (2, -22)$, P_2. Tangents to the parabola with points at P_1, P_2 are then drawn, and intersect at a point Q. What is $m\angle P_1 Q P_2$?

PS. You had better use hide for answers.