**Berkeley Math Tournament , 2018 Spring, Algebra, Geometry, Team , Combinatorics, NT Round**

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– Team Round

**1** A circle with radius $5$ is inscribed in a right triangle with hypotenuse $34$ as shown below. What is the area of the triangle? Note that the diagram is not to scale.

**2** For how many values of $x$ does $20^x \cdot 18^x = 2018^x$?

**3** $2018$ people (call them $A, B, C, \ldots$) stand in a line with each permutation equally likely. Given that $A$ stands before $B$, what is the probability that $C$ stands after $B$?

**4** Consider a standard ($8$-by-$8$) chessboard. Bishops are only allowed to attack pieces that are along the same diagonal as them (but cannot attack along a row or column). If a piece can attack another piece, we say that the pieces threaten each other. How many bishops can you place a chessboard without any of them threatening each other?

**5** How many integers can be expressed in the form: $\pm 1 \pm 2 \pm 3 \pm 4 \pm \cdots \pm 2018$?

**6** A rectangular prism with dimensions $20$ cm by $1$ cm by $7$ cm is made with blue $1$ cm unit cubes. The outside of the rectangular prism is coated in gold paint. If a cube is chosen at random and rolled, what is the probability that the side facing up is painted gold?

**7** Suppose there are $2017$ spies, each with $\frac{1}{2017}$th of a secret code. They communicate by telephone; when two of them talk, they share all information they know with each other. What is the minimum number of telephone calls that are needed for all 2017 people to know all parts of the code?

**8** Alice is playing a game with $2018$ boxes, numbered $1-2018$, and a number of balls. At the beginning, boxes $1-2017$ have one ball each, and box $2018$ has $2018n$ balls. Every turn, Alice chooses $i$ and $j$ with $i > j$, and moves exactly $i$ balls from box $i$ to box $j$. Alice wins if all balls end up in box $1$. What is the minimum value of n so that Alice can win this game?

**9** Circles $A$, $B$, and $C$ are externally tangent circles. Line $PQ$ is drawn such that $PQ$ is tangent to $A$ at $P$, tangent to $B$ at $Q$, and does not intersect with $C$. Circle $D$ is drawn such that it passes through the centers of $A$, $B$, and $C$. Let $R$ be the point on $D$ furthest from $PQ$. If $A$, $B$, and $C$ have radii $3$, $2$, and $1$, respectively, the area of triangle $PQR$ can be expressed in the form of $a + b\sqrt{c}$, where $a$, $b$, and $c$ are integers with $c$ not divisible by any prime square. What is $a + b + c$?

 

**10** A rectangular prism has three distinct faces of area $24$, $30$, and $32$. The diagonals of each distinct face of the prism form sides of a triangle. What is the triangle's area?

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**11** Ankit, Box, and Clark are playing a game. First, Clark comes up with a prime number less than 100. Then he writes each digit of the prime number on a piece of paper (writing $0$ for the tens digit if he chose a single-digit prime), and gives one each to Ankit and Box, without telling them which digit is the tens digit, and which digit is the ones digit. The following exchange occurs:

1. Clark: There is only one prime number that can be made using those two digits.
2. Ankit: I don't know whether I'm the tens digit or the ones digit.
3. Box: I don't know whether I'm the tens digit or the ones digit.
4. Box: You don't know whether you're the tens digit or the ones digit.
5. Ankit: I don't know whether you're the tens digit or the ones digit.

What was Clark's number?

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**12** Let $f : [0, 1] \to \mathbb{R}$ be a monotonically increasing function such that

$$f\left(\frac{x}{3}\right) = \frac{f(x)}{2}$$

$$f(10x) = 2018 - f(x).$$

If $f(1) = 2018$, find $f\left(\frac{12}{13}\right)$.

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**13** Find the value of

$$\frac{1}{\sqrt{2}^1} + \frac{4}{\sqrt{2}^2} + \frac{9}{\sqrt{2}^3} + \cdots$$

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**14** Let $F_1 = 0$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Compute

$$\sum_{n=1}^{\infty} \frac{\sum_{n=1}^{\infty} F_i}{3^n}.$$

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**15** Let triangle $ABC$ have side lengths $AB = 13$, $BC = 14$, $AC = 15$. Let $I$ be the incenter of $ABC$. The circle centered at $A$ of radius $AI$ intersects the circumcircle of $ABC$ at $H$ and $J$. Let $L$ be a point that lies on both the incircle of $ABC$ and line $HJ$. If the minimal possible value of $AL$ is $\sqrt{n}$, where $n \in \mathbb{Z}$, find $n$.

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**–** Algebra Round

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**1** An airplane accelerates at $10$ mph per second and decelerates at $15$ mph/sec. Given that its takeoff speed is $180$ mph, and the pilots want enough runway length to safely decelerate to a stop from any speed below takeoff speed, what's the shortest length that the runway can be allowed to be? Assume the pilots always use maximum acceleration when accelerating. Please give your answer in miles.

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**2** If there is only $1$ complex solution to the equation $8x^3 + 12x^2 + kx + 1 = 0$, what is $k$?

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**3** If $f$ is a polynomial, and $f(-2) = 3$, $f(-1) = -3 = f(1)$, $f(2) = 6$, and $f(3) = 5$, then what is the minimum possible degree of $f$?

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**4** Find

$$\sum_{i=1}^{2016} i(i+1)(i+2) \pmod{2018}.$$

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**5** Find the product of all values of $d$ such that $x^3 + 2x^2 + 3x + 4 = 0$ and $x^2 + dx + 3 = 0$ have a common root.

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**6** Let $x, y, z \in \mathbb{R}$ and $7x^2 + 7y^2 + 7z^2 + 9xyz = 12$. The minimum value of $x^2 + y^2 + z^2$ can be expressed as $\frac{a}{b}$ where $a, b \in \mathbb{Z}, \gcd(a, b) = 1$. What is $a + b$?

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**7** Let

$$h_n := \sum_{k=0}^{n} \binom{n}{k} \frac{2^{k+1}}{(k+1)}.$$

Find

$$\sum_{n=0}^{\infty} \frac{h_n}{n!}.$$

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**8** Compute $\displaystyle\sum_{k=1}^{1009} (-1)^{k+1} \binom{2018-k}{k-1} 2^{2019-2k}$.

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**9** Suppose

$$\frac{1}{3}\frac{(x+1)(x-3)}{(x+2)(x-4)} + \frac{1}{4}\frac{(x+3)(x-5)}{(x+4)(x-6)} - \frac{2}{11}\frac{(x+5)(x-7)}{(x+6)(x-8)} = \frac{53}{132}.$$

Also, suppose $x > 0$. Then $x$ can be written as $a + \sqrt{b}$ where $a$ and $b$ are integers. Find $a + b$.

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**10** Let $a$,$b$,$c$ be the roots of the equation $x^3 - 2018x + 2018 = 0$. Let $q$ be the smallest positive integer for which there exists an integer $p,\ 0 < p \leq q$, such that

$$\frac{a^{p+q} + b^{p+q} + c^{p+q}}{p+q} = \left(\frac{a^p + b^p + c^p}{p}\right)\left(\frac{a^q + b^q + c^q}{q}\right).$$

Find $p^2 + q^2$.

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**Tie 1** A train accelerates at $10$ mph/min, and decelerates at $20$ mph/min. The train's maximum speed is $300$ mph. What's the shortest amount of the time that the train could take to travel $500$ miles, if it has to be stationary at both the start and end of its trip? Please give your answer in minutes.

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**Tie 2** Suppose $2$ cars are going into a turn the shape of a half-circle. Car $1$ is traveling at $50$ meters per second and is hugging the inside of the turn, which has radius $200$ meters. Car $2$ is trying to pass Car $1$ going along the turn, but in order to do this, he has to move to the outside of the turn, which has radius $210$. Suppose that both cars come into the turn side by side, and that they also end the turn being side by side. What was the average speed of Car $2$, in meters per second, throughout the turn?

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**Tie 3** Find

$$\sum_{k=0}^{k=672} \binom{2018}{3k+2} \pmod 3$$

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**–** Geometry Round

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**1** A cube has side length $5$. Let $S$ be its surface area and $V$ its volume. Find $\frac{S^3}{V^2}$ .

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**2** A $1$ by $1$ square $ABCD$ is inscribed in the circle $m$. Circle $n$ has radius $1$ and is centered around $A$. Let $S$ be the set of points inside of $m$ but outside of $n$. What is the area of $S$?

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**3** If $A$ is the area of a triangle with perimeter $1$, what is the largest possible value of $A^2$?

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**4** There are six lines in the plane. No two of them are parallel and no point lies on more than three lines. What is the minimum possible number of points that lie on at least two lines?

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**5** A point is picked uniformly at random inside of a square. Four segments are then drawn in connecting the point to each of the vertices of the square, cutting the square into four triangles. What is the probability that at least two of the resulting triangles are obtuse?

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**6** A triangle $T$ has all integer side lengths and at most one of its side lengths is greater than ten. What is the largest possible area of $T$ ?

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**7** A line in the $xy$-plane has positive slope, passes through the point $(x, y) = (0, 29)$, and lies tangent to the ellipse defined by $\frac{x^2}{100} + \frac{y^2}{400} = 1$. What is the slope of the line?

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**8** What is the largest possible area of a triangle with largest side length $39$ and inradius $10$?

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**9** What is the least integer a greater than $14$ so that the triangle with side lengths $a - 1$, $a$, and $a + 1$ has integer area?

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**10** A plane cuts a sphere of radius $1$ into two pieces, one of which has three times the surface area of the other. What is the area of the disk that the sphere cuts out of the plane?

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**Tie 1** Line segment $\overline{AE}$ of length $17$ bisects $\overline{DB}$ at a point $C$. If $\overline{AB} = 5$, $\overline{BC} = 6$ and $\angle BAC = 78^o$ degrees, calculate $\angle CDE$.

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**Tie 2** Points $A, B, C$ are chosen on the boundary of a circle with center $O$ so that $\angle BAC$ encloses an arc of $120$ degrees. Let $D$ be chosen on $\overline{BA}$ so that $\angle AOD$ is a right angle. Extend $\overline{CD}$ so that it intersects with $O$ again at point $P$. What is the measure of the arc, in degrees, that is enclosed by $\angle ACP$? Please use the $tan^{-1}$ function to express your answer.

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**Tie 3** Consider a regular polygon with $2^n$ sides, for $n \geq 2$, inscribed in a circle of radius $1$. Denote the area of this polygon by $A_n$. Compute $\prod_{i=2}^{\infty} \frac{A_i}{A_{i+1}}$

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**–** Combinatorics Round

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**1** Bob has $3$ different fountain pens and $11$ different ink colors. How many ways can he fill his fountain pens with ink if he can only put one ink in each pen?

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**2** At the Berkeley Math Tournament, teams are composed of $6$ students, each of whom pick two distinct subject tests out of $5$ choices. How many different distributions across subjects are possible for a team?

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**3** Consider the $9 \times 9$ grid of lattice points $\{(x, y)|0 \leq x, y \leq 8\}$. How many rectangles with nonzero area and sides parallel to the $x, y$ axes are there such that each corner is one of the lattice points and the point $(4, 4)$ is not contained within the interior of the rectangle? ($(4, 4)$ is allowed to lie on the boundary of the rectangle).

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**4** Alice starts with an empty string and randomly appends one of the digits $2$, $0$, $1$, or $8$ until the string ends with $2018$. What is the probability Alice appends less than $9$ digits before stopping?

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**5** Alice and Bob play a game where they start from a complete graph with $n$ vertices and take turns removing a single edge from the graph, with Alice taking the fi rst turn. The first player to disconnect the graph loses. Compute the sum of all $n$ between $2$ and $100$ inclusive such that

Alice has a winning strategy. (A complete graph is one where there is an edge between every pair of vertices.)

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**6** Compute

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{i}3^{-(i+j)}.$$

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**7** Let S be the set of line segments between any two vertices of a regular $21$-gon. If we select two distinct line segments from $S$ at random, what is the probability they intersect? Note that line segments are considered to intersect if they share a common vertex.

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**8** Moor and nine friends are seated around a circular table. Moor starts out holding a bottle, and whoever holds the bottle passes it to the person on his left or right with equal probability until everyone has held the bottle. Compute the expected distance between Moor and the last person to receive the bottle, where distance is the fewest number of times the bottle needs to be passed in order to go back to Moor.

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**9** Let $S$ be the set of integers from $1$ to $13$ inclusive. A permutation of $S$ is a function $f : S \to S$ such that $f(x) \neq f(y)$ if $x \neq y$. For how many distinct permutations $f$ does there exists an $n$ such that $f^n(i) = 13 - i + 1$ for all $i$.

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**10** Consider a $2 \times n$ grid where each cell is either black or white, which we attempt to tile with $2 \times 1$ black or white tiles such that tiles have to match the colors of the cells they cover. We first randomly select a random positive integer $N$ where $N$ takes the value $n$ with probability $\frac{1}{2^n}$. We then take a $2 \times N$ grid and randomly color each cell black or white independently with equal probability. Compute the probability the resulting grid has a valid tiling.

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**Tie 1** Every face of a cube is colored one of $3$ colors at random. What is the expected number of edges that lie along two faces of different colors?

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**Tie 2** $6$ people stand in a circle with water guns. Each person randomly selects another person to shoot. What is the probability that no pair of people shoots at each other?

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**Tie 3** Alice and Bob are playing rock paper scissors. Alice however is cheating, so in each round, she has a $\frac{3}{5}$ chance of winning, $\frac{2}{5}$ chance of drawing, and $\frac{2}{5}$ chance of losing. The first person to win $5$ more rounds than the other person wins the match. What is the probability Alice wins?

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**–** Number Theory

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**1** How many multiples of $20$ are also divisors of $17!$?

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**2** Suppose for some positive integers, that $\frac{p+\frac{1}{q}}{q+\frac{1}{p}} = 17$. What is the greatest integer $n$ such that $\frac{p+q}{n}$ is always an integer?

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**3** Find the minimal $N$ such that any $N$-element subset of $\{1, 2, 3, 4, ..., 7\}$ has a subset $S$ such that the sum of elements of $S$ is divisible by $7$.

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**4** What is the remainder when $201820182018...$ [$2018$ times] is divided by $15$?

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**5** If ri are integers such that $0 \le r_i < 31$ and $r_i$ satis fies the polynomial $x^4 + x^3 + x^2 + x \equiv 30$ (mod $31$), find

$$\sum_{i=1}^{4}(r_i^2+1)^{-1} \quad (mod \;\; 31)$$

where $x^{-1}$ is the modulo inverse of $x$, that is, it is the unique integer $y$ such that $0 < y < 31$ and $xy - 1$ is divisible by $31$.

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**6** Ankit wants to create a pseudo-random number generator using modular arithmetic. To do so he starts with a seed $x_0$ and a function $f(x) = 2x + 25$ (mod $31$). To compute the $k$-th pseudo random number, he calls $g(k)$ de ned as follows:

$$g(k) = \begin{cases} x_0 & \text{if } \; k = 0 \\ f(g(k-1)) & \text{if } \; k > 0 \end{cases}$$

If $x_0$ is $2017$, compute $\sum_{j=0}^{2017} g(j)$ (mod $31$).

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**7** Determine the number of ordered triples $(a, b, c)$, with $0 \le a, b, c \le 10$ for which there exists $(x, y)$ such that $ax^2 + by^2 \equiv c$ (mod $11$)

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**8** How many $1 < n \le 2018$ such that the set

$$\{0, 1, 1 + 2, ..., 1 + 2 + 3 + ... + i, ..., 1 + 2 + ... + n - 1\}$$

is a permutation of $\{0, 1, 2, 3, 4, ..., ; n - 1\}$ when reduced modulo $n$?

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**9** Compute the following:

$$\sum_{x=0}^{99}(x^2+1)^{-1} \quad (mod \;\; 199)$$

where $x^{-1}$ is the value $0 \le y \le 199$ such that $xy - 1$ is divisible by $199$.

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**10** Evaluate the following

$$\prod_{j=1}^{50}\left(2cos\left(\frac{4\pi j}{101}\right)+1\right).$$

**Tie 1** Compute the least positive $x$ such that $25x - 6$ is divisible by $1001$.

**Tie 2** An integer $a$ is a quadratic nonresidue modulo a prime $p$ if there does not exist $x \in Z$ such that $x^2 \equiv a$ (mod $p$). How many ordered pairs $(a, b)$ modulo $29$ exist such that

$$a + b \equiv 1 \pmod{29}$$

where both $a$ and $b$ are quadratic nonresidues modulo $29$?

**Tie 3** Let $f : Z^2 \to C$ be a function such that $f(x+11, y) = f(x, y+11) = f(x, y)$, and $f(x, y)f(z, w) = f(xz - yw, xw + yz)$. How many possible values can $f(1, 1)$ have?