Berkeley Math Tournament , 2019 Spring, Algebra, Geometry, Discrete, Individual + Team Round

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–	Algebra Round
1	How many integers x satisfy $x^2 - 9x + 18 < 0$?
2	Find the point p in the first quadrant on the line $y = 2x$ such that the distance between p and p', the point reflected across the line $y = x$, is equal to $\sqrt{32}$.
3	There are several pairs of integers (a, b) satisfying $a^2 - 4a + b^2 - 8b = 30$. Find the sum of the sum of the coordinates of all such points.
4	Two real numbers x and y are both chosen at random from the closed interval $[-10, 10]$. Find the probability that $x^2 + y^2 < 10$. Express your answer as a common fraction in terms of π.
5	Find the sum of all real solutions to $(x^2 - 10x - 12)^{x^2+5x+2} = 1$
6	Find the maximum value of $\frac{x}{y}$ if x and y are real numbers such that $x^2 + y^2 - 8x - 6y + 20 = 0$.
7	Let r_1, r_2, r_3 be the (possibly complex) roots of the polynomial $x^3 + ax^2 + bx + \frac{4}{3}$. How many pairs of integers a, b exist such that $r_1^3 + r_2^3 + r_3^3 = 0$?
8	A biased coin has a $\frac{6 + 2\sqrt{3}}{12}$ chance of landing heads, and a $\frac{6 - 2\sqrt{3}}{12}$ chance of landing tails. What is the probability that the number of times the coin lands heads after being flipped 100 times is a multiple of 4? The answer can be expressed as $\frac{1}{4} + \frac{1 + a^b}{c \cdot d^e}$ where a, b, c, d, e are positive integers. Find the minimal possible value of $a + b + c + d + e$.
9	Let a_n be the product of the complex roots of $x^{2n} = 1$ that are in the first quadrant of the complex plane. That is, roots of the form $a + bi$ where $a, b > 0$. Let $r = a_1 \cdots a_2 \cdot \ldots \cdot a_{10}$. Find the smallest integer k such that r is a root of $x^k = 1$.
10	Find the number of ordered integer triplets x, y, z with absolute value less than or equal to 100 such that $2x^2 + 3y^2 + 3z^2 + 2xy + 2xz - 4yz < 5$.
Tie 1	Compute the maximum real value of a for which there is an integer b such that $\frac{ab^2}{a+2b} = 2019$. Compute the maximum possible value of a.

Tie 2 If P is a function such that $P(2x) = 2^{-3}P(x) + 1$, find $P(0)$.

Tie 3 There are two equilateral triangles with a vertex at $(0, 1)$, with another vertex on the line $y = x+1$ and with the final vertex on the parabola $y = x^2+1$. Find the area of the larger of the two triangles.

– Geometry Round

1 Consider the figure (attached), where every small triangle is equilateral with side length 1. Compute the area of the polygon $AEKS$.
`https://cdn.artofproblemsolving.com/attachments/c/7/671748fe0fce7b8f89532ca66011d119f9b7a png`
Posted for the link in the related post collection (`https://artofproblemsolving.com/community/c2503497_2019_bmt_spring`), with the figure

2 A set of points in the plane is called *full* if every triple of points in the set are the vertices of a non-obtuse triangle. What is the largest size of a full set?

3 Let $ABCD$ be a parallelogram with $BC = 17$. Let M be the midpoint of $\overline{BC}$ and let N be the point such that $DANM$ is a parallelogram. What is the length of segment $\overline{NC}$?

4 The area of right triangle ABC is 4, and the length of hypotenuse AB is 12. Compute the perimeter of $\triangle ABC$.

5 Find the area of the set of all points z in the complex plane that satisfy $|z - 3i| + |z - 4| \le 5\sqrt{2}$.

6 Let $\triangle ABE$ be a triangle with $\frac{AB}{3} = \frac{BE}{4} = \frac{EA}{5}$. Let $D \neq A$ be on line $\overline{AE}$ such that $AE = ED$ and D is closer to E than to A. Moreover, let C be a point such that $BCDE$ is a parallelogram. Furthermore, let M be on line $\overline{CD}$ such that $\overline{AM}$ bisects $\angle BAE$, and let P be the intersection of $\overline{AM}$ and $\overline{BE}$. Compute the ratio of PM to the perimeter of $\triangle ABE$.

7 Points A, B, C, D are vertices of an isosceles trapezoid, with $\overline{AB}$ parallel to $\overline{CD}$, $AB = 1$, $CD = 2$, and $BC = 1$. Point E is chosen uniformly and at random on $\overline{CD}$, and let point F be the point on $\overline{CD}$ such that $EC = FD$. Let G denote the intersection of $\overline{AE}$ and $\overline{BF}$, not necessarily in the trapezoid. What is the probability that $\angle AGB > 30^\circ$?

8 Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let G denote the centroid of $\triangle ABC$, and let G_A denote the image of G under a reflection across $\overline{BC}$, with G_B the image of G under a reflection across $\overline{AC}$, and G_C the image of G under a reflection across $\overline{AB}$. Let O_G be the circumcenter of $\triangle G_AG_BG_C$ and let X be the intersection of $\overline{AO_G}$ with $\overline{BC}$. Let Y denote the intersection of $\overline{AG}$ with $\overline{BC}$. Compute XY.

9 Let $ABCD$ be a tetrahedron with $\angle ABC = \angle ABD = \angle CBD = 90^\circ$ and $AB = BC$. Let E, F, G be points on $\overline{AD}$, BD, and $\overline{CD}$, respectively, such that each of the quadrilaterals $AEFB$, $BFGC$, and $CGEA$ have an inscribed circle. Let r be the smallest real number such that $\frac{[\triangle EFG]}{[\triangle ABC]} \le r$ for all such configurations A, B, C, D, E, F, G. If r can be expressed as $\frac{\sqrt{a - b\sqrt{c}}}{d}$ where a, b, c, d are positive integers with $\gcd(a, b)$ squarefree and c squarefree, find $a + b + c + d$.

Note: Here, $[P]$ denotes the area of polygon P. (This wasn't in the original test; instead they used the notation area(P), which is clear but frankly cumbersome. :P)

10 A *3-4-5 point* of a triangle ABC is a point P such that the ratio $AP : BP : CP$ is equivalent to the ratio $3 : 4 : 5$. If $\triangle ABC$ is isosceles with base $BC = 12$ and $\triangle ABC$ has exactly one $3-4-5$ point, compute the area of $\triangle ABC$.

Tie1 We inscribe a circle ω in equilateral triangle ABC with radius 1. What is the area of the region inside the triangle but outside the circle?

Tie 2 Define the *inverse* of triangle ABC with respect to a point O in the following way: construct the circumcircle of ABC and construct lines AO, BO, and CO. Let A' be the other intersection of AO and the circumcircle (if AO is tangent, then let $A' = A$). Similarly define B' and C'. Then $A'B'C'$ is the inverse of ABC with respect to O. Compute the area of the inverse of the triangle given in the plane by $A(-6, -21)$, $B(-23, 10)$, $C(16, 23)$ with respect to $O(1, 3)$.

Tie 3 We say that a quadrilateral Q is *tangential* if a circle can be inscribed into it, i.e. there exists a circle C that does not meet the vertices of Q, such that it meets each edge at exactly one point. Let N be the number of ways to choose four distinct integers out of $\{1, ..., 24\}$ so that they form the side lengths of a tangential quadrilateral. Find the largest prime factor of N.

– Discrete Round

1 A fair coin is repeatedly flipped until 2019 consecutive coin flips are the same. Compute the probability that the first and last flips of the coin come up differently.

2 Sylvia has a bag of 10 coins. Nine are fair coins, but the tenth has tails on both sides. Sylvia draws a coin at random from the bag and flips it without looking. If the coin comes up tails, what is the probability that the coin she drew was the 2-tailed coin?

3 There are 15 people at a party; each person has 10 friends. To greet each other each person hugs all their friends. How many hugs are exchanged at this party?

4 There exists one pair of positive integers a, b such that $100 > a > b > 0$ and $\frac{1}{a} + \frac{1}{b} = \frac{2}{35}$. Find $a + b$.

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5 Let $2^{1110} \equiv n \bmod 1111$ with $0 \leq n < 1111$. Compute n.

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6 Define $f(n) = \frac{n^2 + n}{2}$. Compute the number of positive integers n such that $f(n) \leq 1000$ and $f(n)$ is the product of two prime numbers.

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7 (My problem. :D)
Call the number of times that the digits of a number change from increasing to decreasing, or vice versa, from the left to right while ignoring consecutive digits that are equal the *flux* of the number. For example, the flux of 123 is 0 (since the digits are always increasing from left to right) and the flux of 12333332 is 1, while the flux of 9182736450 is 8. What is the average value of the flux of the positive integers from 1 to 999, inclusive?

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8 For a positive integer n, define $\phi(n)$ as the number of positive integers less than or equal to n that are relatively prime to n. Find the sum of all positive integers n such that $\phi(n) = 20$.

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9 Let $z = \frac{1}{2}(\sqrt{2} + i\sqrt{2})$. The sum

$$\sum_{k=0}^{13} \frac{1}{1 - ze^{k \cdot \frac{i\pi}{7}}}$$

can be written in the form $a - bi$. Find $a + b$.

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10 Let $S(n)$ be the sum of the squares of the positive integers less than and coprime to n. For example, $S(5) = 1^2 + 2^2 + 3^2 + 4^2$, but $S(4) = 1^2 + 3^2$. Let $p = 2^7 - 1 = 127$ and $q = 2^5 - 1 = 31$ be primes. The quantity $S(pq)$ can be written in the form

$$\frac{p^2q^2}{6}\left(a - \frac{b}{c}\right)$$

where a, b, and c are positive integers, with b and c coprime and $b < c$. Find a.

Tie 1 Compute the probability that a random permutation of the letters in BERKELEY does not have the three E's all on the same side of the Y.

Tie 2 Find the sum of first two integers $n > 1$ such that 3^n is divisible by n and $3^n - 1$ is divisible by $n - 1$.

Tie 3 Let $\{a, b, c, d, e, f, g, h\}$ be a permutation of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. What is the probability that $\overline{abc} + \overline{def}$ is even?

– Team Round

1 Find the maximum integral value of k such that $0 \leq k \leq 2019$ and $|e^{2\pi i \frac{k}{2019}} - 1|$ is maximal.

2 Find the remainder when 2^{2019} is divided by 7.

3 A cylinder with radius 5 and height 1 is rolling on the (unslanted) floor. Inside the cylinder, there is water that has constant height $\frac{15}{2}$ as the cylinder rolls on the floor. What is the volume of the water?

4 Let C be the number of ways to arrange the letters of the word CATALYSIS, T be the number of ways to arrange the letters of the word TRANSPORT, S be the number of ways to arrange the letters of the word STRUCTURE, and M be the number of ways to arrange the letters of the word MOTION. What is $\frac{C-T+S}{M}$?

5 What is the minimum distance between $(2019, 470)$ and $(21a - 19b, 19b + 21a)$ for $a, b \in Z$?

6 At a party, 2019 people decide to form teams of three. To do so, each turn, every person not on a team points to one other person at random. If three people point to each other (that is, A points to B, B points to C, and C points to A), then they form a team. What is the probability that after $65,536$ turns, exactly one person is not on a team

7 How many distinct ordered pairs of integers (b, m, t) satisfy the equation $b^8+m^4+t^2+1 = 2019$?

8 Let (k_i) be a sequence of unique nonzero integers such that $x^2 - 5x + k_i$ has rational solutions. Find the minimum possible value of

$$\frac{1}{5}\sum_{i=1}^{\infty}\frac{1}{k_i}$$

9 You wish to color every vertex, edge, face, and the interior of a cube one color each such that no two adjacent objects are the same color. Faces are adjacent if they share an edge. Edges are adjacent if they share a vertex. The interior is adjacent to all of its faces, edges, and vertices. Each face is adjacent to all of its edges and vertices. Each edge is adjacent to both of its vertices. What is the minimum number of colors required to do this?

10 Compute the remainder when the product of all positive integers less than and relatively prime to 2019 is divided by 2019.

11 A baseball league has 64 people, each with a different 6-digit binary number whose base-10 value ranges from 0 to 63. When any player bats, they do the following: for each pitch, they swing if their corresponding bit number is a 1, otherwise, they decide to wait and let the ball pass. For

example, the player with the number 11 has binary number 001011. For the first and second pitch, they wait; for the third, they swing, and so on. Pitchers follow a similar rule to decide whether to throw a splitter or a fastball, if the bit is 0, they will throw a splitter, and if the bit is 1, they will throw a fastball.
If a batter swings at a fastball, then they will score a hit; if they swing on a splitter, they will miss and get a "strike." If a batter waits on a fastball, then they will also get a strike. If a batter waits on a splitter, then they get a "ball." If a batter gets 3 strikes, then they are out; if a batter gets 4 balls, then they automatically get a hit. For example, if player 11 pitched against player 6 (binary is 000110), the batter would get a ball for the first pitch, a ball for the second pitch, a strike for the third pitch, a strike for the fourth pitch, and a hit for the fifth pitch; as a result, they will count that as a "hit." If player 11 pitched against player 5 (binary is 000101), however, then the fifth pitch would be the batter's third strike, so the batter would be "out."
Each player in the league plays against every other player exactly twice; once as batter, and once as pitcher. They are then given a score equal to the number of outs they throw as a pitcher plus the number of hits they get as a batter. What is the highest score received?

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12 2019 people (all of whom are perfect logicians), labeled from 1 to 2019, partake in a paintball duel. First, they decide to stand in a circle, in order, so that Person 1 has Person 2 to his left and person 2019 to his right. Then, starting with Person 1 and moving to the left, every person who has not been eliminated takes a turn shooting. On their turn, each person can choose to either shoot one non-eliminated person of his or her choice (which eliminates that person from the game), or deliberately miss. The last person standing wins. If, at any point, play goes around the circle once with no one getting eliminated (that is, if all the people playing decide to miss), then automatic paint sprayers will turn on, and end the game with everyone losing. Each person will, on his or her turn, always pick a move that leads to a win if possible, and, if there is still a choice in what move to make, will prefer shooting over missing, and shooting a person closer to his or her left over shooting someone farther from their left. What is the number of the person who wins this game? Put "0" if no one wins.

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13 Triangle $\triangle ABC$ has $AB = 13$, $BC = 14$, and $CA = 15$. $\triangle ABC$ has incircle γ and circumcircle ω. γ has center at I. Line AI is extended to hit ω at P. What is the area of quadrilateral $ABPC$?

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14 A regular hexagon has positive integer side length. A laser is emitted from one of the hexagon's corners, and is reflected off the edges of the hexagon until it hits another corner. Let a be the distance that the laser travels. What is the smallest possible value of a^2 such that $a > 2019$?

You need not simplify/compute exponents.

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15 A group of aliens from Gliese 667 Cc come to Earth to test the hypothesis that mathematics is indeed a universal language. To do this, they give you the following information about their mathematical system:

- For the purposes of this experiment, the Gliesians have decided to write their equations in the same syntactic format as in Western math. For example, in Western math, the expression "$5+4$" is interpreted as running the "$+$" operation on numbers 5 and 4. Similarly, in Gliesian math, the expression $\alpha\gamma\beta$ is interpreted as running the "γ" operation on numbers α and β.

• You know that γ and η are the symbols for addition and multiplication (which works the same in Gliesian math as in Western math), but you don't know which is which. By some bizarre coincidence, the symbol for equality is the same in Gliesian math as it is in Western math; equality is denoted with an "$=$" symbol between the two equal values.

• Two symbols that look exactly the same have the same meaning. Two symbols that are different have different meanings and, therefore, are not equal.

They then provide you with the following equations, written in Gliesian, which are known to be true:
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–	Individual Round
1	Let p be a polynomial with degree less than 4 such that $p(x)$ attains a maximum at $x = 1$. If $p(1) = p(2) = 5$, find $p(10)$.
2	Let A, B, C be unique collinear points$AB = BC = \frac{1}{3}$. Let P be a point that lies on the circle centered at B with radius $\frac{1}{3}$ and the circle centered at C with radius $\frac{1}{3}$. Find the measure of angle $\angle PAC$ in degrees.
3	If $f(x + y) = f(xy)$ for all real numbers x and y, and $f(2019) = 17$, what is the value of $f(17)$?
4	Justin is being served two different types of chips, A-chips, and B-chips. If there are 3 B-chips and 5 A-chips, and if Justin randomly grabs 3 chips, what is the probability that none of them are A-chips?
5	Point P is $\sqrt{3}$ units away from plane A. Let Q be a region of A such that every line through P that intersects A in Q intersects A at an angle between 30^o and 60^o . What is the largest possible area of Q?
6	How many square inches of paint are needed to fully paint a regular 6-sided die with side length 2 inches, except for the $\frac{1}{3}$-inch diameter circular dots marking 1 through 6 (a different number per side)? The paint has negligible thickness, and the circular dots are non-overlapping.
7	Let $\triangle ABC$ be an equilateral triangle with side length M such that points E_1 and E_2 lie on side AB, F_1 and F_2 lie on side BC, and $G1$ and $G2$ lie on side AC, such that $$m = \overline{AE_1} = \overline{BE_2} = \overline{BF_1} = \overline{CF_2} = \overline{CG_1} = \overline{AG_2}$$

and the area of polygon $E_1E_2F_1F_2G_1G_2$ equals the combined areas of $\triangle\ AE_1G_2$, $\triangle\ BF_1E_2$, and $\triangle\ CG_1F_2$. Find the ratio $\frac{m}{M}$.
https://cdn.artofproblemsolving.com/attachments/a/0/88b36c6550c42d913cdddd4486a3dde251327
png

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8 Let $\phi = \frac{1}{2019}$. Define

$$g_n = \begin{cases} 0 & \text{if } \ round(n\phi) = round \ \ ((n-1)\phi) \\ 1 & \text{otherwise.} \end{cases}.$$

where round (x) denotes the round function.
Compute the expected value of g_n if n is an integer chosen from interval $[1, 2019^2]$.

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9 Define an *almost-palindrome* as a string of letters that is not a palindrome but can become a palindrome if one of its letters is changed. For example, $TRUST$ is an almost-palindrome because the R can be changed to an S to produce a palindrome, but $TRIVIAL$ is not an almost-palindrome because it cannot be changed into a palindrome by swapping out only one letter (both the A and the L are out of place). How many almost-palindromes contain fewer than 4 letters.

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10 Let $MATH$ be a square with $MA = 1$. Point B lies on AT such that $\angle MBT = 3.5\angle BMT$. What is the area of $\triangle\ BMT$?

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11 A regular 17-gon with vertices $V_1, V_2, ..., V_{17}$ and sides of length 3 has a point P on V_1V_2 such that $V_1P = 1$. A chord that stretches from V_1 to V_2 containing P is rotated within the interior of the heptadecagon around V_2 such that the chord now stretches from V_2 to V_3. The chord then hinges around V_3, then V_4, and so on, continuing until P is back at its original position. Find the total length traced by P.

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12 Box is thinking of a number, whose digits are all "1". When he squares the number, the sum of its digit is 85. How many digits is Box's number?

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13 Two circles O_1 and O_2 intersect at points A and B. Lines $\overline{AC}$ and $\overline{BD}$ are drawn such that C is on O_1 and D is on O_2 and $\overline{AC} \perp \overline{AB}$ and $\overline{BD} \perp \overline{AB}$. If minor arc $AB = 45$ degrees relative to O_1 and minor arc $AB = 60$ degrees relative to O_2 and the radius of $O_2 = 10$, the area of quadrilateral $CADB$ can be expressed in simplest form as $a + b\sqrt{k} + c\sqrt{\ell}$. Compute $a + b + c + k + \ell$.

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14 On a 24 hour clock, there are two times after $01:00$ for which the time expressed in the form $hh:mm$ and in minutes are both perfect squares. One of these times is $01:21$, since 121 and $60 + 21 = 81$ are both perfect squares. Find the other time, expressed in the form $hh:mm$.

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15 How many distinct positive integers can be formed by choosing their digits from the string 04072019?

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16 Let ABC be a triangle with $AB = 26$, $BC = 51$, and $CA = 73$, and let O be an arbitrary point in the interior of $\triangle ABC$. Lines ℓ_1, ℓ_2, and ℓ_3 pass through O and are parallel to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. The intersections of ℓ_1, ℓ_2, and ℓ_3 and the sides of $\triangle ABC$ form a hexagon whose area is A. Compute the minimum value of A.

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17 Let C be a circle of radius 1 and O its center. Let $\overline{AB}$ be a chord of the circle and D a point on $\overline{AB}$ such that $OD = \frac{\sqrt{2}}{2}$ such that D is closer to A than it is to B, and if the perpendicular line at D with respect to $\overline{AB}$ intersects the circle at $E$and F, $AD = DE$. The area of the region of the circle enclosed by $\overline{AD}$, $\overline{DE}$, and the minor arc AE may be expressed as $\frac{a+b\sqrt{c}+d\pi}{e}$ where a, b, c, d, e are integers, gcd $(a, b, d, e) = 1$, and c is squarefree. Find $a + b + c + d + e$

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18 Define $f(x, y)$ to be $\frac{|x|}{|y|}$ if that value is a positive integer, $\frac{|y|}{|x|}$ if that value is a positive integer, and zero otherwise. We say that a sequence of integers ℓ_1 through ℓ_n is *good* if $f(\ell_i, \ell_{i+1})$ is nonzero for all i where $1 \le i \le n-1$, and the score of the sequence is $\sum_{i=1}^{n-1} f(\ell_i, \ell_{i+1})$

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19 Let a and b be real numbers such that $\max_{0\le x\le 1} |x^3 - ax - b|$ is as small as possible. Find $a + b$ in simplest radical form.

(Hint: If $f(x) = x^3 - cx - d$, then the maximum (or minimum) of $f(x)$ either occurs when $x = 0$ and/or $x = 1$ and/or when x satisfies $3x^2 - c = 0$).

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20 Define a sequence F_n such that $F_1 = 1$, $F_2 = x$, $F_{n+1} = xF_n + yF_{n-1}$ where and x and y are positive integers. Suppose $\frac{1}{F_k} = \sum_{n=1}^{\infty} \frac{F_n}{d^n}$ has exactly two solutions (d, k) with $d > 0$ is a positive integer. Find the least possible positive value of d.

Tie 1 Let p be a prime and n a positive integer below 100. What's the probability that p divides n?

Tie 2 The origami club meets once a week at a fixed time, but this week, the club had to reschedule the meeting to a different time during the same day. However, the room that they usually meet has 5 available time slots, one of which is the original time the origami club meets. If at any given time slot, there is a 30 percent chance the room is not available, what is the probability the origami club will be able to meet at that day?

Tie 3 Ankit, Bill, Charlie, Druv, and Ed are playing a game in which they go around shouting numbers in that order. Ankit starts by shouting the number 1. Bill adds a number that is a factor of the number of letters in his name to Ankit's number and shouts the result. Charlie does the same with Bill's number, and so on (once Ed shouts a number, Ankit does the same procedure to Ed's number, and the game goes on). What is the sum of all possible numbers that can be the 23rd shout?

Tie 4 Consider a regular triangular pyramid with base $\triangle ABC$ and apex D. If we have $AB = BC = AC = 6$ and $AD = BD = CD = 4$, calculate the surface area of the circumsphere of the pyramid.

Tie 5 Ankit, Box, and Clark are taking the tiebreakers for the geometry round, consisting of three problems. Problem k takes each k minutes to solve. If for any given problem there is a $\frac{1}{3}$ chance for each contestant to solve that problem first, what is the probability that Ankit solves a problem first?